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DEVELOPMENT OF AN ADVANCED CONTINUUM
THEORY FOR COMPOSITE LAMINATES

Phase II Annual Report

Volume I

March 31, 1992

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FOR COMPOSITE LAMINATES

Phase II Annual Report

Volume I

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G. R. Ghanimati



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VOLUME II (Attachments)

1.0 INTRODUCTION

1.1 Significance of the Problem

Advanced materials for aerospace, structural, power and propulsion applications offer significant advantages in terms of efficiency and cost. This has been the reason for the ever increasing use of advanced composite materials in recent years. For example, most new or recently manufactured aircraft include structural parts made of composite materials. Although practical usage of composite materials in the modern age originated in aircraft and aerospace industries, their advantages are now recognized by all major industries. A widespread and efficient application of composite materials requires detailed and reliable knowledge of their physical properties and, in turn, of their behavior under applied loads. There are a number of important technical problems associated with the use of composite materials. One such problem is the effect of discontinuities (holes and notches) on the strength of composite laminates. This issue is critical for the determination of the load bearing capacity of composite laminates; which is directly applicable to the design of composite panels and the location of fastener holes. Indeed, the manufacture and repair of advanced composite structures have serious problems connected with the placement of fastener holes. This is especially relevant to composite panel repair, both in the field and at the repair facility. At the present time all depots are confronted with these problems. The lack of appropriate data has resulted in new and in-service designs which are often unnecessarily conservative and expensive (both in cost and turn-around-time). Another related problem concerning composite materials is the issue of interlaminar response of composite materials which is directly related to delamination and edge effects in composites. In recent time, delamination has become the most feared failure mode in laminated composite structures. It can exhibit unstable crack growth, and while delamination failure itself is not usually a catastrophic event, it can perpetrate such a condition due to its weakening influence on a component in its resistance to subsequent failure modes. What had begun in 1970 as somewhat of an academic curiosity turned into a beehive

of research activity in recent years. Study of delamination is now one of the most prominent topics in composite mechanics research. Another related issue in the engineering application of composite materials is the use of numerical methods and, in particular, the finite element method in linear/nonlinear modeling of composite laminates. The various composite shell and composite solid elements that are available today are not adequate for advanced applications. These elements are formulated using one or another form of classical shell theories and although they provide acceptable results in simple loading conditions, but they are not capable of predicting accurate structural response for extreme in-use loading conditions of aerospace systems. One reason for this deficiency has been the lack of powerful theory for composite laminates. Yet another issue in composite laminates is the development of constitutive relations which adequately represent the effect of the constituents. In general, our knowledge of the thermo-mechanical behavior of composite materials and their constitutive relations has lagged behind advances in such other related areas as increasingly sophisticated computers and computational methods for solution of complex problems. Proper understanding and characterization of available and new composite laminates and implementation of their constitutive models in modern (computational) solution procedures is a vital component for safe, economical and competitive design in aerospace applications. It is believed that for a reliable structural analysis, the constitutive models of composite materials should be based on sound thermo-mechanical principles. All the foregoing problems share one common deficiency, namely, the lack of an adequate and sound theory applicable to composite laminates, a theory predicated upon solid physical principles and sound mathematical analysis that can account for the effects of micro-structure, material nonlinearity, geometric nonlinearity, interlaminar stresses, and complex geometry.

In phase I of this research project the feasibility of the development of such a theory was studied. The present work is a continuation of the phase I effort to develop a complete thermo-mechanical theory for composite laminates that exhibits the following characteristics:

- a) It accounts for the effects of micro-structure.
- b) It accounts for the effects of geometric nonlinearity.
- c) It accounts for the effects of curvature.
- d) It accounts for the effects of interlaminar stresses. The three components of the interlaminar stresses are included in the theory and can be determined numerically.
- e) It has a continuum character.
- f) It is applicable to both static and dynamic problems.

Due to the use of Cosserat surface theory, we have called the proposed theory *Cosserat composite theory*.

In the past, several theories have been proposed for the modeling of multilayered plates and shells. Noor and Burton (1990) assessed the computational models for multilayered composite shells. They presented a list of 400 related publications and they identified the following four general approaches for constructing two-dimensional theories for multilayered shells:

1. method of hypotheses.
2. method of expansion.
3. asymptotic integration technique.
4. iterative methods and methods of successive corrections.

Some of these approaches were reviewed in the phase I of this research project. Although each of these theories has its own advantages, we would like to emphasize that, as a general assessment, none of these theories possesses the features of *Cosserat composite theory* collectively. In particular, the interlaminar stresses were not addressed in most of these theories and their applications were limited to situations where the composite laminate is made of a few number of layers. These limitations are not present in the *Cosserat composite theory*.

The *Cosserat composite theory*, because of its continuum character, is best suited for modeling thick composite laminates composed of several (order of hundreds) thin plies. Furthermore, the interlaminar stresses, which are responsible for delamination failures, are incorporated into the formulation of the theory in a natural and consistent manner and without any ad hoc assumptions. Considering these technical issues, the objectives of our research project were developed. In the following a detailed description of these objectives is presented.

1.2 Objectives of the Present Research Project

Some technical issues associated with the application of composite laminates were addressed above. The present research effort is directed toward studying these issues. In this regard the following objectives have been identified to be addressed in the project:

1. Development of strain measures based on the kinematics of micro- and macro-structures of composite laminates and relating these strain measures to the composite stress and composite stress couple tensors. In particular, development of constitutive relations for elastic behavior of composite laminates in the context of infinitesimal deformations that can be applicable to any loading condition including pure bending and pure extension is a part of this task.
2. Further development of the *Cosserat composite theory* to a thermo-mechanical theory for multi-constituent, anisotropic laminates. The theory outlined in phase I of the project is a nonlinear continuum theory which is applicable to composite laminates that are composed of two homogeneous and isotropic constituents within the context of purely mechanical theory. The generalization of the theory to include the effect of temperature, anisotropy and in particular stacking sequence/fiber orientation in fiber reinforced composite laminates will be carried out in this task.

3. Application of the theory to various practical problems. The micro-macro continuum structure built in the kinematics of *Cosserat composite theory* provides an ideal model for the analysis of thick composite laminates composed of a large number of thin plies. Due to the importance of the interlaminar stresses and edge effects at the boundaries of composite plates and shells the *Cosserat composite theory* will be utilized for the analysis of composite laminates under in-plane and out-of-plane loading conditions. The existing theories that address the interlaminar stresses are not manageable when the number of layers exceeds seven or eight (Pagano, 1989). We commented earlier on the significance of the effect of discontinuity in composite laminates and the urgent need for solutions to such problems. In this regard, the analysis of an initially flat composite laminate containing a circular hole will be pursued and the effects of the discontinuity will be studied.

4. Variational formulation of the *Cosserat composite theory* suitable for finite element discretization of composite laminates. This part includes the following activities:

- formulation of field equations and constitutive relations in a form suitable for finite element implementation and derivation of weak forms of these equations.
- linearization of the weak form in terms of linearized strain measures and finite element interpolation of the representative Cosserat surface and the director field and approximation of various stress components.
- numerical implementation of the theory and evaluation of performance of the proposed finite element model in solving various practical problems.

Our efforts during the year 1991 were directed toward the achievement of these objectives. The results are summarized next.

1.3 Present Status of the Project

The results of our research efforts during the last year are presented in section 2.0 through section 7.0 of this report. A summary of the contents of these sections is presented in the following.

In section 2.0 the kinematics of the micro- and macro-structures were examined and the relationship between strain measures at micro- and macro-levels were derived. The field equations for composite laminates were derived through a direct integration of field equations of classical continuum mechanics. The linearized kinematic measures were derived in the context of infinitesimal deformation and the relation of linear strain measures with displacement vector and director displacement vector were obtained. The equations of motion in the linear theory were derived and were presented for both curved and flat geometries.

Section 3.0 showed the derivation of constitutive relations for composite laminates. A procedure for deriving the relation between composite quantities (i.e., *composite stress tensor* and *composite couple stress tensor*) and strain measures at macro-level were presented. The derivation was performed for a bi-constituent composite laminate and the constitutive relations were expressed in terms of material constants associated with every individual layer.

Section 4.0 presented the complete theory for linear elastic composite laminates. The relationship between the displacement vector and the director displacement vector was derived based on the geometrical continuity at interfaces. The field equations were derived in terms of displacement vector and it was shown that classical continuum theory can be derived from *Cosserat composite theory* for the case of a single constituent. The theory was further simplified for bi-laminate micro-structure composed of isotropic constituents. Finally the constitutive relations for composite stress tensor, composite couple stress tensor and interlaminar stress vector were derived in terms of the displacement vector, its gradients and material constants of individual constituents.

Section 5.0 was the extension of the theory for multi-constituent composites. The micro-structure or representative element was assumed to be composed of several constituents which repeated themselves in the layering direction. The development of this section is particularly suited for fiber reinforced composites where the fiber direction changes in the stacking sequence of the plies. The theory was simplified for the case of isotropic constituents.

Section 6.0 presented the extension of the theory from a purely mechanical theory to a thermo-mechanical theory. In this section composite field quantities corresponding to the heat flux vector, the heat supply and the specific entropy of classical thermo-mechanical theory were introduced and the equation of local balance of energy and the Clausius-Duhem inequality were derived in terms of these composite field quantities.

Section 7.0 presented the constitutive relations of linear thermo-elastcity for composite laminates. These constitutive relations were derived for the composite stress tensor, composite couple stress tensor, entropy, heat flux vector and heat flux couple vector. The developments of this section were parallel to those of section 4.0 and a set of coupled thermo-mechanical field equations in terms of the displacement vector and the temperature were presented. The continuation of these efforts is briefly discussed in the Future Work section.

We would like to emphasize that because of the very complex nature of composite materials, any general continuum theory capable of representing various behavioral aspects of these materials will be complicated. During the course of these developments, two lines of thought regarding the interpretation of various field quantities of the theory were considered. Accordingly, some of the earlier developments of the phase I were re-examined and a few modifications regarding the definition of the field quantities were introduced. This process required a concentrated effort by all team members, but it had its own rewards. The modeling and the mathematical issues were clarified and a better understanding of various features of the theory was achieved.

The field equations for composite laminates, as presented in section 2.0 and section 6.0, were derived directly from corresponding field equations of classical continuum mechanics. In this derivation the existence of a micro-structure in the form of a Cosserat surface was assumed for the composite laminates. A parallel development of the theory is presented in Volume II of the report in which the conservation laws of a single micro-structure (i.e., a Cosserat surface) were integrated across the layering direction of the composite laminate. This integration process provided the global balance laws for composite laminates and the governing field equations were derived from these global balance laws.

Some of these developments were presented in the AFOSR contractors meeting on Mechanics of Materials in October 1991 in Dayton, Ohio. A copy of this presentation is included in Volume II. Also the draft of a technical paper presenting the field equations of composite laminates as derived from conservation laws was prepared for publication in the International Journal of Engineering Science. A copy of this draft report is also included in Volume II of the present report. The research efforts presented here were carried out by Dr. M. Panahandeh, Dr. G. R. Ghanimati, Dr. V. Schricker and Dr. M. Mahzoon, a visiting scholar with the University of California, Berkeley.

1.4 Future Work

We are very encouraged with the developments of this project. These developments provide a framework for the modeling and analysis of various composite laminates and can be extended to study a wide range of technical issues associated with the application of composite laminates such as nonlinear material behavior, stability analysis, damage growth and failure mechanisms, to name a few.

The continuation of the present work to achieve the technical objectives of the project, as outlined in Section 1.2, is planned for this year. In particular, the analysis of composite laminates under in-plane and out-of-plane loadings and the numerical solution of the system of

equations derived in Section 4.0 is given immediate priority. Theoretical development of the theory for cylindrical and spherical geometries is underway. The derivation of constitutive relations for orthotropic materials presenting fiber reinforced plies with various fiber orientations and different stacking sequences is also a part of this year's effort. The finite element formulation of the theory and development of a composite shell element will be followed immediately after the completion of the theoretical developments.

2.0 MICRO-MACRO CONTINUUM MODEL OF COMPOSITE LAMINATES

2.1 Kinematics of Micro- and Macro-Structures

Let the points of a region \mathcal{R} in a three dimensional Euclidean space be referred to a fixed right-handed rectangular Cartesian coordinate system x^i ($i = 1,2,3$) and let θ^i ($i = 1,2,3$) be a general *convected* curvilinear coordinate system defined by the transformation $x^i = x^i(\theta^j)$. We assume this transformation is nonsingular in \mathcal{R} . Furthermore, let ξ represent the coordinate of a micro-structure in the layering direction with $\xi = 0$ corresponding to the bottom surface of the micro-structure. We recall that a convected coordinate system is normally defined in relation to a continuous body and moves continuously with the body throughout the motion of the body from one configuration to another.

Throughout this work, all Latin indices (subscripts or superscripts) take the values 1,2,3; all Greek indices (subscripts or superscripts) take the values 1,2 and the usual summation convention is employed. We will use a comma for partial differentiation with respect to coordinates θ^α and a superposed dot for material time derivative, i.e., differentiation with respect to time holding the material coordinates fixed. Also, we use a vertical bar (|) for covariant differentiation. In what follows, when there is a possibility of confusion, quantities which represent the same physical/geometrical concepts will be denoted by the same symbol but with an added asterisk (*) for classical three dimensional continuum mechanics and no addition for composite laminate (macro-structure). For example, the mass densities of a body in the contexts of the classical continuum mechanics, and the composite laminate (macro-structure) will be denoted by ρ^* and ρ , respectively.

The micro-macro continuum model of a composite laminate is illustrated in Figures 1 and 2. Figure 1 shows a typical composite laminate (only three micro-structures are shown in this figure). Figure 2 shows a shell-like micro-structure with its associated coordinates. This micro-structure is composed of two constituents and can be generalized for cases of multi-constituent composites.

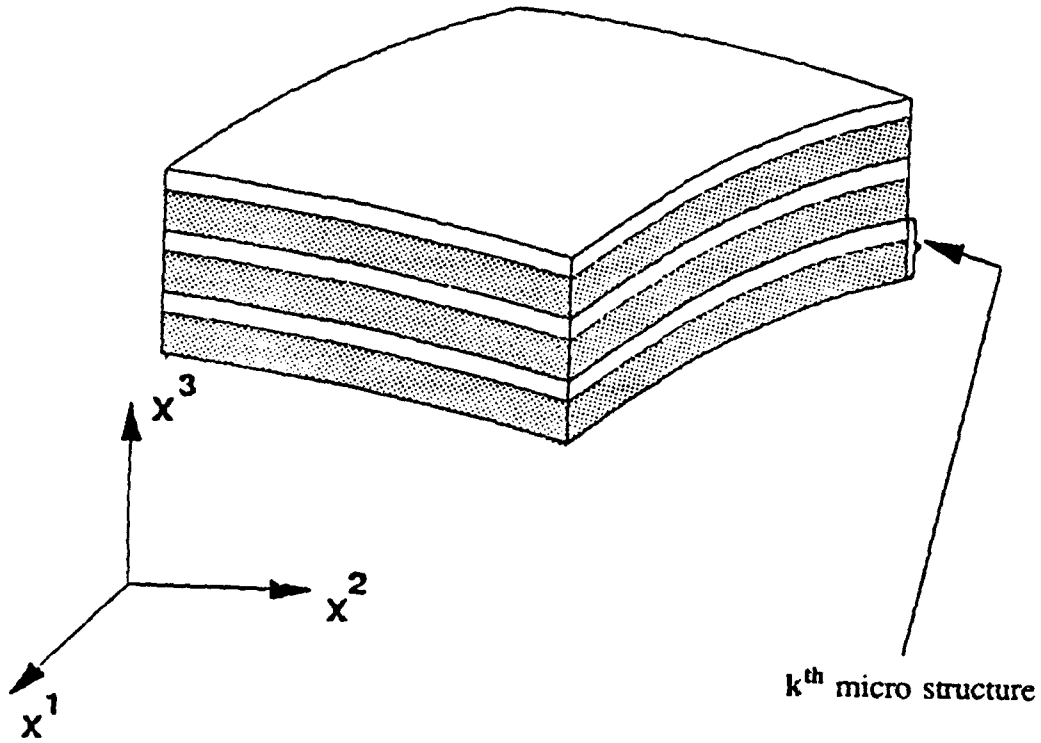


Figure 1
A composite laminate consisting of alternating
layers of two materials

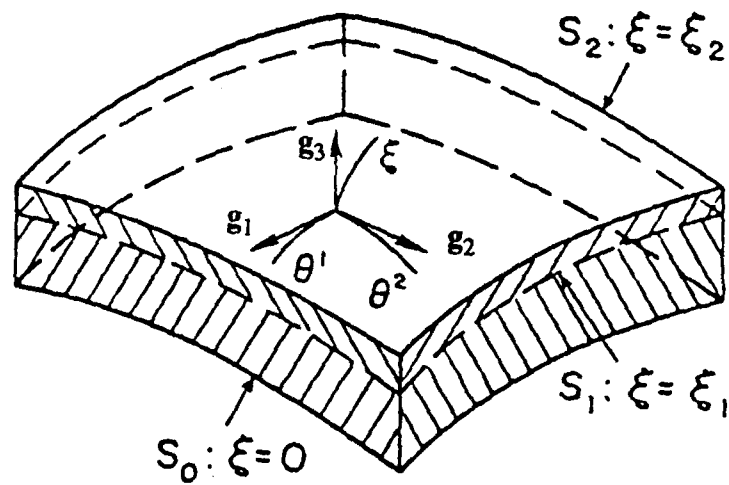


Figure 2
A SHELL-LIKE MICRO-STRUCTURE (REPRESENTATIVE ELEMENT)

We begin the development of the kinematical results by assuming that the position vector of a particle P^* of a representative element (k^{th} micro-structure), i.e., $p^*(\theta^\alpha, \theta^{3(k)}, \xi, t)$ in the present configuration has the form

$$p^* = r(\theta^\alpha, \theta^{3(k)}, t) + \xi d(\theta^\alpha, \theta^{3(k)}, t) \quad (k = 1, \dots, n) \quad (2.1)$$

where r is the position vector for the surface $\xi = 0$ and d is the director field. $\theta^{3(k)}$, at this point, is an identifier for the k^{th} micro-structure. Greek super- or subscripts will assume values of 1 and 2 only. The dual of (2.1) in a reference configuration is given by

$$P^* = R^*(\theta^\alpha, \theta^{3(k)}) + \xi D(\theta^\alpha, \theta^{3(k)}) \quad (2.2)$$

If the reference configuration is taken to be the initial configuration at time $t = 0$, we obtain

$$\begin{aligned} p^*(\theta^\alpha, \theta^{3(k)}, \xi, 0) &= r(\theta^\alpha, \theta^{3(k)}, 0) + \xi d(\theta^\alpha, \theta^{3(k)}, 0) \\ &= R(\theta^\alpha, \theta^{3(k)}) + \xi D(\theta^\alpha, \theta^{3(k)}) = P^*(\theta^\alpha, \theta^{3(k)}, \xi) \end{aligned} \quad (2.3)$$

The velocity vector v^* of the three-dimensional shell-like micro-structure at time t is given by

$$v^* = \frac{\partial p^*(\theta^\alpha, \theta^{3(k)}, \xi, t)}{\partial t} = \dot{p}^*(\theta^\alpha, \theta^{3(k)}, \xi, t) \quad (2.4)$$

where a superposed dot denotes the material time derivative, holding θ^α and ξ fixed. From (2.1) and (2.4) we obtain

$$v^* = v + \xi w \quad (2.5)$$

where

$$\mathbf{v} = \dot{\mathbf{r}} \quad , \quad \mathbf{w} = \dot{\mathbf{d}} \quad (2.6)$$

The base vectors for the micro- and macro-structures are denoted by \mathbf{g}_i^* and \mathbf{g}_i , respectively, and we have

$$\begin{aligned} \mathbf{g}_\alpha^* &= \frac{\partial \mathbf{p}^*}{\partial \theta^\alpha} \quad , \quad \mathbf{g}_3^* = \frac{\partial \mathbf{p}^*}{\partial \xi} \\ \mathbf{g}_\alpha &= \frac{\partial \mathbf{p}^*}{\partial \theta^\alpha} \Big|_{\xi=0} \quad , \quad \mathbf{g}_3 = \frac{\partial \mathbf{p}^*}{\partial \xi} \Big|_{\xi=0} \end{aligned} \quad (2.7)$$

Using (2.1) and (2.7) we obtain the following relations between \mathbf{g}_i^* and \mathbf{g}_i

$$\begin{aligned} \mathbf{g}_\alpha^* &= \mathbf{g}_\alpha + \xi \mathbf{d}_{,\alpha} \\ \mathbf{g}_3^* &= \mathbf{g}_3 = \mathbf{d} \end{aligned} \quad (2.8)$$

where $(\)_{,\alpha}$ denotes partial differentiation with respect to θ^α .

By a smoothing assumption we suggest the existence of continuous vector functions $\mathbf{g}_i(\theta^\alpha, \theta^3)$ for the macro-structure with the following property

$$\mathbf{g}_i(\theta^\alpha, \theta^3) \Big|_{\theta^3 = \theta^{3(k)}} = \mathbf{g}_i(\theta^\alpha, \theta^{3(k)}) \quad (2.9)$$

where $\mathbf{g}(\theta^\alpha, \theta^{3(k)})$ are defined according to (2.7)₂. A similar smoothing assumption is also made for the director \mathbf{d} which we like to attach to every point of the macro-structure. Based on the smoothing assumptions we can write (2.8)₁ as follows

$$\mathbf{g}_\alpha^* = \mathbf{g}_\alpha + \xi \mathbf{g}_k \{ {}_3^k{}_\alpha \} \quad (2.10)$$

where $\{ \ }$ stands for the Christoffel symbol of the second kind and is defined as

$$\{ {}_3^k{}_\alpha \} = g^{kj} [3\alpha, j]_{kj} = \frac{1}{2} g^{kj} \left(\frac{\partial g_{3j}}{\partial \theta^\alpha} + \frac{\partial g_{\alpha j}}{\partial \theta^3} - \frac{\partial g_{3\alpha}}{\partial \theta^j} \right)$$

The following relations can also be derived between the components of metric tensors

$$g_{ij}^* = g_i^* \cdot g_j^* \text{ and } g_{ij} = g_i \cdot g_j$$

$$g_{\alpha\beta}^* = g_{\alpha\beta} + \xi[\{3^k{}_{\alpha}\}g_{\beta k} + \{3^k{}_{\beta}\}g_{\alpha k}] + \xi^2\{3^k{}_{\alpha}\}\{3^j{}_{\beta}\}g_{kj}$$

$$g_{\alpha 3}^* = g_{\alpha 3} + \xi\{3^k{}_{\alpha}\}g_{k 3} \quad (2.11)$$

$$g_{33}^* = g_{33}$$

which after simplification and linearization in terms of ξ reduce to

$$g_{\alpha\beta}^* = g_{\alpha\beta} + \xi g_{\alpha\beta,3}$$

$$g_{\alpha 3}^* = g_{\alpha 3} + \frac{1}{2} \xi g_{33,\alpha} \quad (2.12)$$

$$g_{33}^* = g_{33}$$

The determinants of metric tensors g_{ij}^* and g_{ij} are also related according to the following relation

$$g^* = g + \xi \Delta \quad (2.13)$$

where

$$g^* = \det(g_{ij}^*) \quad , \quad g = \det(g_{ij})$$

$$\Delta = \begin{bmatrix} g_{11,3} & g_{12,3} & g_{33,1} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{bmatrix} + \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{12,3} & g_{22,3} & g_{33,2} \\ g_{13} & g_{23} & g_{33} \end{bmatrix}$$

and the final result has been linearized in terms of ξ .

We recall that the director d is the same as g_3 and therefore when referred to the base vectors g_i it has only one non-zero component, namely $d^3 = 1$, so we can write

$$\mathbf{d} = d^i \mathbf{g}_i, \quad d^\alpha = 0, \quad d^3 = 1 \quad (2.15)$$

$$d_i = g_{ij} d^j, \quad d_i = g_{i3} \quad (i=1,2,3)$$

where d_i and d^i denote the covariant and contravariant components of \mathbf{d} referred to \mathbf{g}^i and \mathbf{g}_i , respectively. The gradient of the director \mathbf{d} may be obtained as follows

$$d_{,i} = g_{3,i} = \{3 \quad k \quad i\} g_k = d^k{}_{,i} g_k$$

The vertical bar (|) denotes covariant differentiation with respect to g_{ij} . For convenience we introduce the notations

$$\lambda_{ij} = \mathbf{g}_i \cdot \mathbf{d}_{,j} = d_{i|j} \quad (2.17)$$

$$\lambda^i{}_j = \mathbf{g}^i \cdot \mathbf{d}_{,j} = d^i{}_{|j}$$

From (2.17) it is clear that

$$\lambda^i{}_j = g^{ik} \lambda_{kj} \quad (2.18)$$

Making use of (2.17) and (2.16) we have:

$$\lambda^i{}_j = d^i{}_{|j} = \{3 \quad i \quad j\} \quad (2.19)$$

$$\lambda_{ij} = g_{ki} \lambda^k{}_j = [3j, i]$$

Consider now the velocity vector \mathbf{v} which can be written in the form

$$\mathbf{v} = v^i \mathbf{g}_i = v_i \mathbf{g}^i \quad (2.20)$$

Again we make a smoothing assumption for the existence of the vector function $\mathbf{v}(\theta^\alpha, \theta^3)$ such that $\mathbf{v}(\theta^\alpha, \theta^3)_{|\theta^3=\theta^{3(k)}} = \dot{\mathbf{r}}(\theta^\alpha, \theta^{3(k)})$ after which we can define the gradient of the velocity field and we have

$$v_{,i} = (v^j \mathbf{g}_j)_{,i} = v^j{}_{|i} \mathbf{g}_j \quad (2.21)$$

We now introduce the notations

$$v_{ij} = g_i \cdot v_j = v_{i|j} \quad (2.22)$$

$$v^i_j = g^i \cdot v_j = v^i_{|j}$$

From (2.22) it is clear that

$$v^i_j = g^{ik} v_{kj} \quad (2.23)$$

$$v_{,i} = v_{ij} g^j = v^j_i g_j$$

We observe that both λ_{ij} and v_{ij} represent the covariant derivative of vector components and hence transform as components of second order covariant tensors.

We may decompose v_{ij} into its symmetric and skew-symmetric parts, i.e.,

$$v_{ij} = v_{(ij)} + v_{[ij]} = \eta_{ij} + \omega_{ij}$$

$$\eta_{ij} = v_{(ij)} = \frac{1}{2} (v_{ij} + v_{ji}) = \eta_{ji} \quad (2.24)$$

$$\omega_{ij} = v_{[ij]} = \frac{1}{2} (v_{ij} - v_{ji}) = -\omega_{ji}$$

Also in view of (2.6), (2.7)₂ and (2.24)₁ we may express \dot{g}_i in the form

$$\dot{g}_\alpha = v_{,\alpha} = (\eta_{k\alpha} + \omega_{k\alpha}) g^k \quad (2.25)$$

$$\dot{g}_3 = \dot{d} = w = w_k g^k = w^k g_k$$

The gradient of the director velocity in θ^α direction is obtained by writing

$$w_{,\alpha} = \dot{d}_{,\alpha} = \overline{(\dot{d}_{,\alpha})} = \overline{\lambda_{\alpha}^k g_k}$$

$$= \overline{\lambda_{\alpha}^k} g_k + \lambda_{\alpha}^k \dot{g}_k = \dot{\lambda}_{\alpha}^k g_k + \lambda_{\alpha}^{\beta} \dot{g}_{\beta} + \lambda_{\alpha}^3 \dot{g}_3$$

$$\begin{aligned}
 &= \dot{\lambda}_{\alpha}^k g_k + \lambda_{\alpha}^{\beta} (\eta_{k\beta} + \omega_{k\beta}) g^k + \lambda_{\alpha}^3 w^k g_k \\
 &= [\dot{\lambda}_{\alpha}^k + \lambda_{\alpha}^{\beta} (\eta_{\beta}^k + \omega_{\beta}^k) + \lambda_{\alpha}^3 w^k] g_k
 \end{aligned} \tag{2.26}$$

The dual of the above expressions in the reference configuration can be written easily by substituting appropriate capital letters for small letters.

We now introduce relative kinematic measures γ_{ij} and κ_{ij} such that

$$\gamma_{ij} = \frac{1}{2} (g_{ij} - G_{ij}) = \gamma_{ji} \tag{2.27}$$

$$\kappa_{ij} = \lambda_{ij} - \Lambda_{ij} \tag{2.28}$$

where

$$G_{ij} = G_i \cdot G_j \tag{2.29}$$

$$\Lambda_{ij} = [3j,i] = \frac{1}{2} (G_{3i,j} + G_{ji,3} - G_{3j,i}) \tag{2.30}$$

Making use of (2.12) and similar expressions for the reference configuration we can relate relative kinematic measures γ_{ij}^* of the micro-structure as follows

$$\begin{aligned}
 \gamma_{\alpha\beta}^* &= \gamma_{\beta\alpha}^* = \frac{1}{2} (g_{\alpha\beta}^* - G_{\alpha\beta}^*) = \frac{1}{2} [(g_{\alpha\beta} + \xi g_{\alpha\beta,3}) - (G_{\alpha\beta} + \xi G_{\alpha\beta,3})] \\
 &= \gamma_{\alpha\beta} + \frac{1}{2} \xi (g_{\alpha\beta,3} - G_{\alpha\beta,3}) \\
 &= \gamma_{\alpha\beta} + \frac{1}{2} \xi (\kappa_{\alpha\beta} + \kappa_{\beta\alpha})
 \end{aligned} \tag{2.31}$$

$$\begin{aligned}
 \gamma_{\alpha 3}^* &= \gamma_{3\alpha}^* = \frac{1}{2} [(g_{\alpha 3} + \frac{1}{2} \xi g_{33,\alpha}) - (G_{\alpha 3} + \frac{1}{2} \xi G_{33,\alpha})] \\
 &= \gamma_{\alpha 3} + \frac{1}{2} \xi \kappa_{3\alpha}
 \end{aligned} \tag{2.32}$$

$$\gamma_{33}^* = \gamma_{33} \quad (2.33)$$

In obtaining the above results we have noted that

$$\lambda_{\alpha\beta} = [3\beta, \alpha] = \frac{1}{2} (g_{3\alpha, \beta} - g_{3\beta, \alpha}) + \frac{1}{2} g_{\alpha\beta, 3}$$
$$\lambda_{3\alpha} = [3\alpha, 3] = \frac{1}{2} g_{33, \alpha} \quad (2.34)$$

$$\kappa_{\alpha\beta} + \kappa_{\beta\alpha} = g_{\alpha\beta, 3} - G_{\alpha\beta, 3}$$

and we have linearized the result in terms of ξ .

2.2 Basic Field Equations for Micro- and Macro-Structures

The three-dimensional equations of motion in classical continuum mechanics are recorded here for the k^{th} representative element (micro-structure) in the present configuration

$$\dot{\rho^* g^{*1/2}} = 0 \quad (2.35)$$

$$\mathbf{T}^{*i}_{,j} + \rho^* \mathbf{b}^* g^{*1/2} = \rho^* \dot{\mathbf{v}}^* g^{*1/2} \quad (2.36)$$

$$\mathbf{g}_i^* \times \mathbf{T}^{*i} = 0 \quad (2.37)$$

where

$$\mathbf{t}^* = g^{*-1/2} \mathbf{T}^{*i} n_i^* \quad , \quad \mathbf{T}^{*i} = g^{*1/2} \tau^{*ij} \mathbf{g}_j^* \quad (2.38)$$

The argument of all starred functions recorded above is $(\theta^\alpha, \theta^{3(k)}, \xi, t)$ and the equations are written for each and every representative element ($k = 1, 2, \dots, n$) which is assumed to repeat itself in the present model.

Now introduce the following quantities for each micro-structure:

Composite Stress Vector \mathbf{T}^i :

$$\mathbf{T}^i(\theta^\alpha, \theta^{3(k)}, t) \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} \mathbf{T}^{*i}(\theta^\alpha, \theta^{3(k)}, \xi, t) d\xi \quad (2.39)$$

Composite Stress Couple Vector \mathbf{S}^α :

$$\mathbf{S}^\alpha(\theta^\alpha, \theta^{3(k)}, t) \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} \xi \mathbf{T}^{*\alpha}(\theta^\alpha, \theta^{3(k)}, \xi, t) d\xi \quad (2.40)$$

Composite Mass Density ρ :

$$\rho g^{1/2} \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} d\xi \quad (2.41)$$

$$\rho g^{1/2}(z^\alpha) \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} (\xi)^\alpha \rho^* g^{*1/2} d\xi \quad (\alpha=1,2) \quad (2.42)$$

Composite Body Force Density b:

$$\rho g^{1/2} b \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* b^* g^{*1/2} d\xi \quad (2.43)$$

Composite Body Couple Density c:

$$\rho g^{1/2} c \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* b^* g^{*1/2} \xi d\xi \quad (2.44)$$

The quantities on the left-hand side of equations (2.39)-(2.44) are discrete in terms of the variable $\theta^{3(k)}$ which are made continuous by smoothing assumptions. The composite mass density ρ_0 in the reference configuration is also defined as follows:

$$\rho_0 G^{1/2} = \frac{1}{\xi_2} \int_0^{\xi_2} \rho_0^* G^{*1/2} d\xi \quad (2.45)$$

where ρ_0^* is the mass density of the micro-structure in the reference configuration. Since $\rho^* g^{*1/2} = \rho_0^* G^{*1/2}$, the continuity equation for the macro-structure is readily seen to be

$$\rho g^{1/2} = \rho_0 G^{1/2} \quad (2.46)$$

Now consider equation (2.36) and first divide it by ξ_2 and then integrate with respect to ξ from 0 to ξ_2 to obtain the equation for balance of linear momentum for the macro-structure

$$\begin{aligned} \frac{1}{\xi_2} \int_0^{\xi_2} T^{*\alpha}{}_{,\alpha} d\xi + \frac{1}{\xi_2} \int_0^{\xi_2} \frac{\partial T^{*3}}{\partial \xi} d\xi + \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* b^* g^{*1/2} d\xi \\ = \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* (\dot{v} + \xi w) g^{*1/2} d\xi \end{aligned} \quad (2.47)$$

Each term in the above equation can be represented in terms of the quantities defined in (2.39)-(2.44) except the second term which is the difference between interlaminar stresses above and below the representative element divided by its thickness ξ_2 as

$$\frac{1}{\xi_2} \int_0^{\xi_2} \frac{\partial T^{*3}}{\partial \xi} d\xi = \frac{1}{\xi_2} [T^{*3}(\theta^\alpha, \theta^{3(k+1)}, t) - T^{*3}(\theta^\alpha, \theta^{3(k)}, t)] \quad (2.43)$$

Now we postulate the existence of the continuous vector function $\sigma(\theta^\alpha, \theta^3, t)$ whose values at $\theta^3 = \theta^{3(k)}$ are the same as interlaminar stresses $T^{*3}(\theta^\alpha, \theta^{3(k)}, t)$ and further approximate (2.43) as the gradient of this function in the θ^3 direction. With this in mind we write (2.47) as

$$T^{\alpha}_{,\alpha} + \frac{\partial \sigma}{\partial \theta^3} + \rho b g^{1/2} = \rho g^{1/2}(\dot{v} + z^1 \dot{w}) \quad (2.44)$$

To obtain the equation for balance of director momentum, (2.36) is multiplied by ξ , integrated from 0 to ξ_2 and divided by ξ_2 to get

$$\begin{aligned} \frac{1}{\xi_2} \int_0^{\xi_2} \xi T^{\alpha}_{,\alpha} d\xi + \frac{1}{\xi_2} \int_0^{\xi_2} \xi \frac{\partial T^{*3}}{\partial \xi} d\xi + \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* b^* g^{*1/2} \xi d\xi \\ = \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} (\xi \dot{v} + \xi^2 \dot{w}) d\xi \end{aligned} \quad (2.50)$$

Again the second term in the above equation can be written as

$$\frac{1}{\xi_2} \int_0^{\xi_2} \xi \frac{\partial T^{*3}}{\partial \xi} d\xi = \frac{1}{\xi_2} [\xi T^{*3}]_0^{\xi_2} - \frac{1}{\xi_2} \int_0^{\xi_2} T^{*3} d\xi = \sigma - T^3 \quad (2.51)$$

As a result we have

$$S^{\alpha}_{,\alpha} + \sigma - T^3 + \rho g^{1/2} c = \rho g^{1/2} (z^1 \dot{v} + z^2 \dot{w}) \quad (2.52)$$

which is the equation for balance of director momentum.

Next, we consider (2.37), divide it by ξ_2 and integrate with respect to ξ from $\xi=0$ to $\xi=\xi_2$ and making use of (2.8)_{1,2} we get

$$\frac{1}{\xi_2} \int_0^{\xi_2} (\mathbf{g}_\alpha^* \times \mathbf{T}^{*\alpha} + \mathbf{g}_3^* \times \mathbf{T}^{*3}) d\xi = 0$$

or

(2.53)

$$\frac{1}{\xi_2} \int_0^{\xi_2} (\mathbf{g}_\alpha + \xi \mathbf{d}_{,\alpha}) \times \mathbf{T}^{*\alpha} d\xi + \frac{1}{\xi_2} \int_0^{\xi_2} \mathbf{d} \times \mathbf{T}^{*3} d\xi = 0$$

and substituting from (2.39) and (2.40) we obtain

$$\mathbf{g}_\alpha \times \mathbf{T}^\alpha + \mathbf{d}_{,\alpha} \times \mathbf{S}^\alpha + \mathbf{d} \times \mathbf{T}^3 = 0 \quad (2.54)$$

which can also be written as

$$\mathbf{g}_i \times \mathbf{T}^i + \mathbf{d}_{,\alpha} \times \mathbf{S}^\alpha = 0 \quad (2.55)$$

This is the balance of moment of momentum for the macro-structure.

Now we proceed to obtain an expression for the specific mechanical energy. Such an expression for each micro-structure can be written as

$$\rho^* \mathbf{g}^{*1/2} \dot{\epsilon}^* = \mathbf{T}^{*i} \cdot \mathbf{v}_{,i}^* \quad (2.56)$$

First, using (2.5) we write this equation as

$$\begin{aligned} \rho^* \mathbf{g}^{*1/2} \dot{\epsilon}^* &= \mathbf{T}^{*\alpha} \cdot (\mathbf{v} + \xi \mathbf{w})_{,\alpha} + \mathbf{T}^{*3} \cdot \frac{\partial}{\partial \xi} (\mathbf{v} + \xi \mathbf{w}) \\ &= \mathbf{T}^{*\alpha} \cdot \mathbf{v}_{,\alpha} + \xi \mathbf{T}^{*\alpha} \cdot \mathbf{w}_{,\alpha} + \mathbf{T}^{*3} \cdot \mathbf{w} \end{aligned} \quad (2.57)$$

Dividing (2.57) by ξ_2 and integrating with respect to ξ from $\xi = 0$ to $\xi = \xi_2$ will result in

$$\frac{1}{\xi_2} \int_0^{\xi_2} \rho^* \mathbf{g}^{*1/2} \dot{\epsilon}^* d\xi = \frac{1}{\xi_2} \int_0^{\xi_2} \mathbf{T}^{*\alpha} d\xi \cdot \mathbf{v}_{,\alpha} + \frac{1}{\xi_2} \int_0^{\xi_2} \xi \mathbf{T}^{*\alpha} d\xi \cdot \mathbf{w}_{,\alpha} + \frac{1}{\xi_2} \int_0^{\xi_2} \mathbf{T}^{*3} d\xi \cdot \mathbf{w} \quad (2.58)$$

We now define *composite specific mechanical energy* for the representative element by

$$\rho g^{1/2} \epsilon = \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} \epsilon^* d\xi \quad (2.59)$$

From this definition, the equation of continuity and other definitions (2.39) through (2.44) for composite quantities, (2.58) can be rewritten as

$$\rho g^{1/2} \dot{\epsilon} = T^\alpha \cdot v_{,\alpha} + S^\alpha \cdot w_{,\alpha} + T^3 \cdot w \quad (2.60)$$

Since $v = \dot{r}$, $v_{,\alpha} = (\dot{r})_{,\alpha} = \dot{\bar{v}}_{,\alpha} = \dot{g}_\alpha$ and $w = \dot{d} = \dot{g}_3$, we can further reduce (2.60) to

$$\rho g^{1/2} \dot{\epsilon} = T^i \cdot \dot{g}_i + S^\alpha \cdot w_{,\alpha} \quad (2.61)$$

which is the appropriate expression for the specific mechanical energy of the macro-structure.

2.3 Field Equations in Component Form

We obtained the following field equations for the macro-structure (balance of mass is not recorded since it is a scalar equation)

$$T^{\alpha}_{,\alpha} + \frac{\partial \sigma}{\partial \theta^3} + \rho g^{1/2} b = \rho g^{1/2} (\dot{v} + z^1 \dot{w}) \quad (2.62)$$

$$S^{\alpha}_{,\alpha} + \sigma - T^3 + \rho g^{1/2} c = \rho g^{1/2} (z^1 \dot{v} + z^3 \dot{w}) \quad (2.63)$$

$$g_i \times T^i + d_{,\alpha} \times S^{\alpha} = 0 \quad (2.64)$$

And also the following expression was derived for the specific mechanical energy

$$\rho g^{1/2} \dot{\epsilon} = T^i \cdot \dot{g}_i + S^{\alpha} \cdot w_{,\alpha} \quad (2.65)$$

By referring various vector quantities to the base g_i we would like to write the above equations in component form. First write

$$T^i = g^{1/2} \tau^{ij} g_j \quad (2.66)$$

$$\sigma = \sigma^j g_j \quad (2.67)$$

$$S^{\alpha} = g^{1/2} S^{\alpha j} g_j \quad (2.68)$$

$$b = b^j g_j, \quad c = c^j g_j \quad (2.69)$$

where τ^{ij} and $S^{\alpha j}$ are contravariant components of *composite stress tensor* and *composite couple stress tensor*, respectively, σ^i is the interlaminar stress. Now substitute in (2.62) and obtain

$$(g^{1/2} \tau^{\alpha j} g_j)_{,\alpha} + \frac{\partial}{\partial \theta^3} (\sigma^j g_j) + \rho g^{1/2} b^j g_j = \rho g^{1/2} (\dot{v}^j + z^1 \dot{w}^j) g_j$$

$$(g^{1/2} \tau^{\alpha j})_{,\alpha} g_j + g^{1/2} \tau^{\alpha j} \{j^k{}_{\alpha}\} g_k + \sigma^j_{,3} g_j + \sigma^j \{j^k{}_{,3}\} g_k + \rho g^{1/2} b^j g_j$$

$$= \rho g^{1/2} (\dot{v}^j + z^1 \dot{w}^j) g_j$$

or

$$\begin{aligned} (g^{1/2} \tau^{\alpha j})_{,\alpha} + g^{1/2} \tau^{\alpha k} \{_{k \ j \ \alpha} \} + \sigma^j_{\ 3} + \sigma^k \{_{k \ j \ 3} \} + \rho g^{1/2} b^j \\ = \rho g^{1/2} (\dot{v}^j + z^1 \dot{w}^j) \end{aligned} \quad (2.70)$$

Equation (2.63) reduces to

$$\begin{aligned} (g^{1/2} S^{\alpha j} g_j)_{,\alpha} + \sigma^j g_j - g^{1/2} \tau^3 j + \rho g^{1/2} c^j g_j \\ = \rho g^{1/2} (z^1 \dot{v}^j + z^2 \dot{w}^j) g_j \end{aligned}$$

or

$$(g^{1/2} S^{\alpha j})_{,\alpha} + g^{1/2} S^{\alpha k} \{_{k \ j \ \alpha} \} + \sigma^j - g^{1/2} \tau^3 j + \rho g^{1/2} c^j = \rho g^{1/2} (z^1 \dot{v}^j + z^2 \dot{w}^j) \quad (2.71)$$

Equation (2.64) can be rewritten as

$$g_i \times (g^{1/2} \tau^{ij} g_j) + \lambda^i_{\ \alpha} g_i \times (g^{1/2} S^{\alpha j} g_j) = 0$$

or

$$g^{1/2} (\tau^{ij} + \lambda^i_{\ \alpha} S^{\alpha j}) g_i \times g_j = 0 \quad (2.72)$$

since $g \neq 0$, $g_i \times g_j = \epsilon_{ijk} g^k$ and ϵ_{ijk} is skew-symmetric we conclude that the quantity in parentheses in (2.72) must be symmetric in i and j . As a result, the conservation of angular momentum in component form is the symmetry of T^{ij} defined by

$$T^{ij} \triangleq \tau^{ij} + \lambda^i_{\ \alpha} S^{\alpha j} \quad (2.73)$$

$$T^{ij} = T^{ji} \quad (2.74)$$

The expression (2.65) for the specific mechanical energy can also be written as

$$\begin{aligned}
 \rho g^{1/2} \dot{\epsilon} &= g^{1/2} \tau^{ij} g_j \cdot \dot{g}_i + g^{1/2} S^{\alpha j} g_j \cdot w_{i|\alpha} g_i \\
 &= g^{1/2} (\tau^{\alpha j} g_j \cdot \dot{g}_\alpha + S^{\alpha j} w_{j|\alpha} + \tau^{3j} g_j \cdot \dot{g}_3) \\
 &= g^{1/2} (\tau^{\alpha j} g_j \cdot v_{,\alpha} + \tau^{3j} g_j \cdot w + S^{\alpha j} w_{j|\alpha}) \\
 &= g^{1/2} (\tau^{\alpha j} v_{j|\alpha} + \tau^{3j} w_j + S^{\alpha j} w_{j|\alpha})
 \end{aligned}$$

We have now the component form of the expression for mechanical power

$$P \triangleq \rho \dot{\epsilon} = \tau^{\alpha j} v_{j|\alpha} + \tau^{3j} w_j + S^{\alpha j} w_{j|\alpha} \quad (2.75)$$

An alternative form for mechanical energy expression is derived in which the rates of relative kinematic measures will appear. Using (2.25)₁ and (2.26), we rewrite (2.75) as

$$\begin{aligned}
 P = \rho \dot{\epsilon} &= \tau^{\alpha j} (\eta_{j\alpha} + \omega_{j\alpha}) + \tau^{3j} w_j + S^{\beta j} [\dot{\lambda}_{j\beta} + \lambda_\beta^\alpha (\eta_{j\alpha} + \omega_{j\alpha}) + \lambda_\beta^3 w_j] \\
 &= (\tau^{\alpha j} + S^{\beta j} \lambda_\beta^\alpha) \eta_{j\alpha} + (\tau^{\alpha j} + S^{\beta j} \lambda_\beta^\alpha) \omega_{j\alpha} + S^{\beta j} \dot{\lambda}_{j\beta} \\
 &\quad + (\tau^{3j} + \lambda_\beta^3 S^{\beta j}) w_j
 \end{aligned} \quad (2.76)$$

Recalling (2.73) and using symmetry of T^{ij} and skew-symmetry of w_{ij} we can write (2.76) as

$$P = T^{\alpha j} \eta_{j\alpha} + S^{\beta j} \dot{\lambda}_{j\beta} + T^{3j} w_j + T^{\alpha 3} \omega_{3\alpha} \quad (2.77)$$

By (2.25)₁ we have

$$\dot{g}_\alpha \cdot g_\beta = \eta_{\beta\alpha} + \omega_{\beta\alpha} \quad , \quad \dot{g}_\beta \cdot g_\alpha = \eta_{\alpha\beta} + \omega_{\alpha\beta} \quad (2.78)$$

Therefore

$$\dot{\overline{g_{\alpha\beta}}} = \dot{g}_\alpha \cdot g_\beta + g_\alpha \cdot \dot{g}_\beta = 2\eta_{\alpha\beta} \quad (2.79)$$

$$\eta_{\alpha\beta} = \frac{1}{2} \dot{\overline{g_{\alpha\beta}}} = \dot{\gamma}_{\alpha\beta} \quad (2.80)$$

In the last result we have used the definition of $\gamma_{\alpha\beta}$ from (2.27). By (2.25)_{1,2} we have

$$\dot{\mathbf{g}}_{\alpha} \cdot \mathbf{g}_3 = \eta_{3\alpha} + \omega_{3\alpha} \quad (2.81)$$

$$\dot{\mathbf{g}}_3 \cdot \mathbf{g}_{\alpha} = w_{\alpha} \quad (2.82)$$

Therefore

$$\dot{\overline{\mathbf{g}}}_{\alpha 3} = \dot{\mathbf{g}}_{\alpha} \cdot \mathbf{g}_3 + \dot{\mathbf{g}}_3 \cdot \mathbf{g}_{\alpha} = \eta_{3\alpha} + \omega_{3\alpha} + w_{\alpha} \quad (2.83)$$

$$\eta_{3\alpha} = \dot{\overline{\mathbf{g}}}_{\alpha 3} - (\omega_{3\alpha} + w_{\alpha}) = 2\dot{\gamma}_{\alpha 3} - (\omega_{3\alpha} + w_{\alpha}) \quad (2.84)$$

Again by (2.25)₂ and (2.27)

$$\omega_3 = \dot{\mathbf{g}}_3 \cdot \mathbf{g}_3 = \frac{1}{2} \dot{\mathbf{g}}_{33} \quad (2.85)$$

$$\dot{\gamma}_{33} = \frac{1}{2} \dot{\mathbf{g}}_{33} = w_3 \quad (2.86)$$

and by (2.28)

$$\dot{\lambda}_{j\beta} = \dot{\kappa}_{j\beta} \quad (2.87)$$

Substituting from (2.80), (2.84), (2.86) and (2.87) in (2.77) we get

$$P = T^{\alpha\beta} \dot{\gamma}_{\alpha\beta} + T^{\alpha 3} (2\dot{\gamma}_{\alpha 3} - (\omega_{3\alpha} + w_{\alpha})) + S^{\beta j} \dot{\kappa}_{j\beta} + T^{3\alpha} w_{\alpha} + T^{33} \dot{\gamma}_{33} + T^{\alpha 3} \omega_{3\alpha} \quad (2.88)$$

which is simplified to

$$P = T^{\alpha\beta} \dot{\gamma}_{\alpha\beta} + 2T^{\alpha 3} \dot{\gamma}_{\alpha 3} + T^{33} \dot{\gamma}_{33} + S^{\beta j} \dot{\kappa}_{j\beta} \quad (2.89)$$

If symmetry of T^{ij} and γ_{ij} is considered we can further simplify (2.89) and obtain

$$P = T^{ij} \dot{\gamma}_{ij} + S^{\beta j} \dot{\kappa}_{j\beta} \quad (2.90)$$

or

$$P = (\tau^{ij} + \lambda_{\alpha}^i S^{\alpha j}) \gamma_{ij} + S^{\alpha j} \kappa_{j\alpha} \quad (2.91)$$

2.4 General Constitutive Assumption for Elastic Composite

At this point we postulate the existence of specific internal energy in purely mechanical theory which depends on relative kinematic measures γ_{ij} and $\kappa_{j\alpha}$ as defined in (2.27) and (2.28)

$$\psi = \hat{\psi}(\gamma_{ij}, \kappa_{j\alpha}) \quad (2.92)$$

$$P = \rho \dot{\psi} \quad (2.93)$$

By usual procedures we obtain from (2.91), (2.92) and (2.93)

$$\tau^{ij} = \rho \left(\frac{\partial \hat{\psi}}{\partial \gamma_{ij}} - \lambda_{\alpha}^i \frac{\partial \hat{\psi}}{\partial \kappa_{j\alpha}} \right) \quad (2.94)$$

$$S^{\alpha j} = \rho \frac{\partial \hat{\psi}}{\partial \kappa_{j\alpha}} \quad (2.95)$$

Now the *composite stress vector* T^i and the *composite couple stress vector* S^{α} from (2.66) and (2.68) will be

$$T^i = \rho g^{1/2} \left(\frac{\partial \hat{\psi}}{\partial \gamma_{ij}} - \lambda_{\alpha}^i \frac{\partial \hat{\psi}}{\partial \kappa_{j\alpha}} \right) g_j \quad (2.96)$$

$$S^{\alpha} = \rho g^{1/2} \left(\frac{\partial \hat{\psi}}{\partial \kappa_{j\alpha}} \right) g_j \quad (2.97)$$

The coefficient $\rho g^{1/2}$ can be replaced by $\rho_0 G^{1/2}$ by taking advantage of the continuity equation. Note that by these constitutive relations for T^i and S^{α} the balance of moment of momentum is identically satisfied.

2.5 Linearized Kinematics

For linearized kinematics let

$$\mathbf{r}(\theta^\alpha, \theta^{3(k)}, t) = \mathbf{R}(\theta^\alpha, \theta^{3(k)}) + \epsilon \mathbf{u}(\theta^\alpha, \theta^{3(k)}, t) \quad (2.98)$$

$$\mathbf{d}(\theta^\alpha, \theta^{3(k)}, t) = \mathbf{D}(\theta^\alpha, \theta^{3(k)}) + \epsilon \delta(\theta^\alpha, \theta^{3(k)}, t) \quad (2.99)$$

$$\mathbf{v} = \dot{\mathbf{r}} = \epsilon \dot{\mathbf{u}}, \quad \mathbf{w} = \dot{\mathbf{d}} = \epsilon \dot{\delta} \quad (2.100)$$

where ϵ is a non-dimensional parameter. The motion of the macro-structure describes infinitesimal deformation if the magnitude of the gradient of the displacement vector $\epsilon \mathbf{u}$ and the magnitude of the director displacement vector $\epsilon \delta$ are of the order of $\epsilon \ll 1$ such that in the following developments we can only retain terms which are linear in ϵ . The base vectors \mathbf{g}_i are found from (2.7)₂ as:

$$\mathbf{g}_\alpha = \mathbf{R}_{,\alpha} + \epsilon \mathbf{u}_{,\alpha} \quad (2.101)$$

$$\mathbf{g}_3 = \mathbf{d} = \mathbf{D} + \epsilon \delta \quad (2.102)$$

The corresponding vectors in reference configuration are:

$$\mathbf{G}_\alpha = \mathbf{R}_{,\alpha}, \quad \mathbf{G}_3 = \mathbf{D} \quad (2.103)$$

We now proceed to obtain the relative kinematic measures γ_{ij} and $\kappa_{i\alpha}$. Using (2.103)₂ and (2.101) together with the definition of $\mathbf{g}_{\alpha\beta}$ and $\mathbf{G}_{\alpha\beta}$ we write

$$\mathbf{g}_{\alpha\beta} = (\mathbf{G}_\alpha + \epsilon \mathbf{u}_{,\alpha}) \cdot (\mathbf{G}_\beta + \epsilon \mathbf{u}_{,\beta}) = \mathbf{G}_{\alpha\beta} + \epsilon (\mathbf{G}_\alpha \cdot \mathbf{u}_{,\beta} + \mathbf{u}_{,\alpha} \cdot \mathbf{G}_\beta) + O(\epsilon^2) \quad (2.104)$$

where $O(\epsilon^2)$ denotes terms of order ϵ^2 in displacement gradient, where

$$\begin{aligned} \mathbf{G}_\alpha \cdot \mathbf{u}_{,\beta} + \mathbf{u}_{,\alpha} \cdot \mathbf{G}_\beta &= \mathbf{G}_\alpha \cdot u^j_{|\beta} \mathbf{g}_j + u^j_{|\alpha} \mathbf{g}_j \cdot \mathbf{G}_\beta \\ &= \mathbf{G}_\alpha \cdot u^\gamma_{|\beta} \mathbf{g}_\gamma + \mathbf{G}_\alpha \cdot u^3_{|\beta} \mathbf{g}_3 + u^\gamma_{|\alpha} \mathbf{g}_\gamma \cdot \mathbf{G}_\beta + u^3_{|\alpha} \mathbf{g}_3 \cdot \mathbf{G}_\beta \end{aligned}$$

$$= u^{\gamma}{}_{|\beta} G_{\alpha} \cdot (G_{\gamma} + \epsilon u_{,\gamma}) + u^3{}_{|\beta} G_{\alpha} \cdot (D + \epsilon \delta) + u^{\gamma}{}_{|\alpha} (G_{\gamma} + \epsilon u_{,\gamma}) \cdot G_{\beta} \\ + u^3{}_{|\alpha} (D + \epsilon \delta) \cdot G_{\beta} \quad (2.105)$$

Retaining terms which are of the order of unity in (2.105) and substituting the result in (2.104) we find

$$\gamma_{\alpha\beta} = \frac{1}{2} (g_{\alpha\beta} - G_{\alpha\beta}) = \frac{1}{2} (u_{\alpha|\beta} + u_{\beta|\alpha}) + \frac{1}{2} (u^3{}_{|\beta} D_{\alpha} + u^3{}_{|\alpha} D_{\beta}) \quad (2.106)$$

Here covariant differentiation is supposed to be performed with respect to the metric G_{ij} of the reference configuration and instead of ϵu we have used u with the same assumptions made for linearization. Similarly we can write

$$g_{\alpha 3} = (G_{\alpha} + \epsilon u_{,\alpha}) \cdot (G_3 + \epsilon \delta) = G_{\alpha 3} + \epsilon (G_{\alpha} \cdot \delta + u_{,\alpha} \cdot G_3) + O(\epsilon^3) \\ = G_{\alpha 3} + \epsilon (\delta_{\alpha} + u_{3|\alpha}) + O(\epsilon^2) \quad (2.107)$$

Again using δ instead of $\epsilon \delta$ with the same interpretation we obtain

$$\gamma_{\alpha 3} = \gamma_{3\alpha} = \frac{1}{2} (g_{\alpha 3} - G_{\alpha 3}) = \frac{1}{2} (\delta_{\alpha} + u_{3|\alpha}) \quad (2.108)$$

To find γ_{33} we write

$$g_{33} = (G_3 + \epsilon \delta) \cdot (G_3 + \epsilon \delta) = G_{33} + 2\epsilon \delta_3 + O(\epsilon^2) \quad (2.109)$$

$$\gamma_{33} = \frac{1}{2} (g_{33} - G_{33}) = \delta_3 \quad (2.110)$$

As for the measures $\kappa_{i\alpha}$ we proceed as follows

$$\lambda_{\alpha\beta} = g_{\alpha} \cdot d_{\beta} = (G_{\alpha} + \epsilon u_{,\alpha}) \cdot (D_{\beta} + \epsilon \delta_{\beta}) = \Lambda_{\alpha\beta} + \epsilon (u^j{}_{|\alpha} g_j \cdot D_{\beta} + G_{\alpha} \cdot \delta^j{}_{|\beta} g_j) + O(\epsilon^2) \quad (2.111)$$

where:

$$\begin{aligned} u^j_{|\alpha} g_i \cdot D_{\beta} &= u^{\gamma}_{|\alpha} (G_{\gamma} + \epsilon u_{,\gamma}) \cdot D_{\beta} + u^3_{|\alpha} (G_3 + \epsilon \delta) \cdot D_{\beta} \\ &= u^{\gamma}_{|\alpha} \Lambda_{\gamma\beta} + u^3_{|\alpha} \Lambda_{3\beta} + O(\epsilon) \end{aligned}$$

$$\begin{aligned} G_{\alpha} \cdot \delta^j_{|\beta} g_j &= \delta^{\gamma}_{|\beta} G_{\alpha} \cdot (G_{\gamma} + \epsilon u_{,\gamma}) + \delta^3_{|\beta} G_{\alpha} \cdot (G_3 + \epsilon \delta) \\ &= \delta_{\alpha|\beta} + \delta^3_{|\beta} D_{\alpha} + O(\epsilon) \end{aligned}$$

Substituting these results in (2.101) and using the definition of $\kappa_{\alpha\beta}$ we get

$$\kappa_{\alpha\beta} = \lambda_{\alpha\beta} - \Lambda_{\alpha\beta} = u^j_{|\alpha} \Lambda_{j\beta} + \delta_{\alpha|\beta} + \delta^3_{|\beta} D_{\alpha} \quad (2.112)$$

Now we obtain an expression for $\kappa_{3\alpha}$

$$\begin{aligned} \lambda_{3\alpha} &= g_3 \cdot d_{,\alpha} = (G_3 + \epsilon \delta) \cdot (D_{,\alpha} + \epsilon \delta_{,\alpha}) \\ \lambda_{3\alpha} &= \Lambda_{3\alpha} + \epsilon (G_3 \cdot \delta_{,\alpha} + \delta \cdot D_{,\alpha}) + O(\epsilon^2) \end{aligned} \quad (2.113)$$

We simplify each term separately

$$\begin{aligned} G_3 \cdot \delta_{,\alpha} &= G_3 \cdot (\delta^j_{|\alpha} g_j) = \delta^{\gamma}_{|\alpha} G_3 \cdot (G_{\gamma} + \epsilon u_{,\gamma}) \\ &\quad + \delta^3_{|\alpha} G_3 \cdot (D + \epsilon \delta) = \delta^{\gamma}_{|\alpha} D_{\gamma} + \delta^3_{|\alpha} D_3 + O(\epsilon) \\ &= \delta^j_{|\alpha} D_j + O(\epsilon) \end{aligned} \quad (2.114)$$

$$\begin{aligned} \delta \cdot D_{,\alpha} &= (\delta^j_{|\alpha} g_j) \cdot (D^k_{|\alpha} G_k) = (\delta^j_{|\alpha} g_j) \cdot D^3_{|\alpha} G_3 \\ &= \Lambda^3_{\alpha} (\delta^{\gamma}_{|\alpha} g_{\gamma} + \delta^3_{|\alpha} g_3) \cdot G_3 \\ &= \Lambda^3_{\alpha} [\delta^{\gamma}_{|\alpha} (G_{\gamma} + \epsilon u_{,\gamma}) + \delta^3_{|\alpha} (G_3 + \epsilon \delta)] \cdot D \\ &= O(\epsilon) + \Lambda^3_{\alpha} \delta^{\gamma}_{|\alpha} D_{\gamma} + \Lambda^3_{\alpha} \delta^3_{|\alpha} D^3 = \Lambda^3_{\alpha} \delta^j_{|\alpha} D_j + O(\epsilon) \end{aligned} \quad (2.115)$$

However, since $D^{\alpha} = 0$ and $D^3 = 1$

$$\delta^j D_j = \delta_j D^j = \delta_3 \quad (2.116)$$

Substituting from (2.114), (2.115) and (2.116) in (2.113) and using previous notation for δ we obtain

$$\kappa_{3\alpha} = \lambda_{3\alpha} - \Lambda_{3\alpha} = \delta^j_{|\alpha} D_j + \Lambda_{\alpha}^3 \delta_3 = \delta_{3|\alpha} + \Lambda_{\alpha}^3 \delta_3 \quad (2.117)$$

To recapitulate the relative kinematic measures in linear theory are:

$$\gamma_{\alpha\beta} = \frac{1}{2} (u_{\alpha|\beta} + u_{\beta|\alpha} + u^3_{|\alpha} D_{\beta} + u^3_{|\beta} D_{\alpha})$$

$$\gamma_{\alpha 3} = \gamma_{3\alpha} = \frac{1}{2} (u_{3|\alpha} + \delta_{\alpha})$$

$$\gamma_{33} = \delta_3 \quad (2.118)$$

$$\kappa_{\alpha\beta} = \Lambda_{\beta}^j u_{j|\alpha} + \delta_{\alpha|\beta} + \delta^3_{|\beta} D_{\alpha}$$

$$\kappa_{3\alpha} = \delta_{3|\alpha} + \Lambda_{\alpha}^3 \delta_3$$

For a composite with initially flat plates we can always choose our base vectors G_i such that $G_{ij} = G^{ij} = \delta_{ij}$ and as a result $D^{\alpha} = D_{\alpha} = 0$ and if we confine ourselves to small deformations, then all Christoffel symbols vanish and covariant differentiations reduce to partial differentiations and equations (2.118) for relative kinematic measures will further reduce to

$$\gamma_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha})$$

$$\gamma_{\alpha 3} = \frac{1}{2} (u_{3,\alpha} + \delta_{\alpha})$$

$$(2.119)$$

$$\gamma_{33} = \delta_3$$

$$\kappa_{\alpha\beta} = \delta_{\alpha,\beta} \quad , \quad \kappa_{3\alpha} = \delta_{3,\alpha}$$

In writing these relations it has been noted that $D_{\alpha} = 0$, $D_3 = 1$ and $\Lambda_{\alpha}^j \equiv 0$.

It is also desirable to find a relation between g and G , determinants of metric tensors in present and reference configurations, in the linear theory. We recall the following relations

$$g^{1/2} = g_1 \times g_2 \cdot g_3 \quad , \quad G^{1/2} = G_1 \times G_2 \cdot G_3$$

$$g_1 \times g_2 = (G_1 + \epsilon u_{,1}) \times (G_2 + \epsilon u_{,2})$$

$$= G_1 \times G_2 + \epsilon(u_1 \times G_2 + G_1 \times u_{,2}) + O(\epsilon^2)$$

$$[g_1 g_2 g_3] = [G_1 \times G_2 + \epsilon(u_1 \times G_2 + G_1 \times u_{,2})] \cdot [G_3 + \epsilon \delta] + O(\epsilon^2)$$

$$= G^{1/2} + \epsilon[G_1 \times G_2 \cdot \delta + u_1 \cdot G_2 \times G_3 + G_3 \times G_1 \cdot u_{,2}] + O(\epsilon^2)$$

$$= G^{1/2} + \epsilon G^{1/2}[G^3 \cdot \delta + u_{,1} \cdot G^1 + G^2 \cdot u_{,2}] + O(\epsilon^2)$$

Retaining terms which are of the order of ϵ and using the previous notations for u and δ we get

$$\left(\frac{g}{G}\right)^{1/2} = 1 + \delta^3 + u^\alpha_{,1\alpha} \quad (2.120)$$

Now the equation for balance of mass will readily reduce to

$$\rho_o = \rho \left(\frac{g}{G}\right)^{1/2} = \rho(1 + \delta^3 + u^\alpha_{,1\alpha}) \quad (2.121)$$

and since in linear theory displacement vector u and director displacement δ satisfy linearity assumptions we obtain

$$\rho = \rho_o(1 - \delta^3 - u^\alpha_{,1\alpha}) \quad (2.122)$$

2.6 Linearized Field Equations

We use pertinent results from linear kinematics and usual procedures for linearization to write the field equations in linear theory. It should be recalled in such analysis that g is replaced by G , ρ by ρ_0 and Christoffel symbols are calculated with respect to G_{ij} . By omitting the details, the linear version of the equations of motion are recorded here

$$(G^{1/2} \tau^{\alpha j})_{,\alpha} + G^{1/2} \tau^{\alpha k} \{k^j{}_{\alpha}\} + \sigma^j_{,3} + \sigma^k \{k^j{}_{,3}\} + \rho_0 G^{1/2} b^j = \rho_0 G^{1/2} (\ddot{u}^j + z^1 \ddot{\delta}^j) \quad (2.123)$$

$$(G^{1/2} S^{\alpha j})_{,\alpha} + G^{1/2} S^{\alpha k} \{k^j{}_{\alpha}\} + \sigma^j - G^{1/2} \tau^{3j} + \rho_0 G^{1/2} c^j = \rho_0 G^{1/2} (z^1 \ddot{u}^j + z^2 \ddot{\delta}^j) \quad (2.124)$$

$$T^{ij} = \tau^{ij} + \Lambda^i_{\alpha} S^{\alpha j} = T^{ji} \quad (2.125)$$

For a composite with initially flat plies further simplification can be made. As mentioned earlier, $G = 1$ and all Christoffel symbols vanish identically. The resulting balance equations for such a situation will be

$$\tau^{\alpha j}_{,\alpha} + \sigma^j_{,3} + \rho_0 b^j = \rho_0 (\ddot{u}^j + z^1 \ddot{\delta}^j) \quad (2.126)$$

$$S^{\alpha j}_{,\alpha} + \sigma^j - \tau^{3j} + \rho_0 c^j = \rho_0 (z^1 \ddot{u}^j + z^2 \ddot{\delta}^j) \quad (2.127)$$

$$T^{ij} = \tau^{ij} = \tau^{ji} \quad (2.128)$$

3.0 CONSTITUTIVE RELATIONS FOR LINEAR ELASTICITY

For the representative micro-structure let

$$\tau_{(\alpha)}^{*ij} = c_{(\alpha)}^{ijkl} \gamma_{kl}^* \quad , \quad \alpha = 1, 2 \quad (3.1)$$

where

$$c_{(\alpha)}^{ijkl} = \begin{cases} c_{(1)}^{ijkl} & 0 < \xi < \xi_1 \\ c_{(2)}^{ijkl} & \xi_1 < \xi < \xi_2 \end{cases} \quad (3.2)$$

and $c_{(\alpha)}^{ijkl}$ ($\alpha = 1, 2$) are material constants in associated layers. Now we proceed to calculate T^i and S^α defined in (2.39) and (2.40). First we recall that $T^{*i} = g^{*1/2} \tau^{*ij} g_j^*$, $g^{*1/2} = g^{1/2} (1 + \frac{\xi}{2} \frac{\Delta}{g})$, $g_\gamma^* = g_\gamma + \xi d_{,\gamma}$, $g_3^* = g_3$ and for brevity we omit the index α in relations (3.1) and (3.2)

$$\begin{aligned} T^i &= \frac{1}{\xi_2} \int_0^{\xi_2} T^{*i} d\xi = \frac{1}{\xi_2} \int_0^{\xi_2} g^{*1/2} \tau^{*ij} g_j^* d\xi \\ &= \frac{1}{\xi_2} \int_0^{\xi_2} g^{*1/2} c^{ijkl} \gamma_{kl}^* g_j^* d\xi = \frac{1}{\xi_2} \int_0^{\xi_2} g^{*1/2} (c^{i\alpha\beta} \gamma_{\alpha\beta}^* + 2c^{ij\alpha 3} \gamma_{\alpha 3}^* + c^{ij33} \gamma_{33}^*) g_j^* d\xi \end{aligned}$$

Substitute from (2.31), (2.32) and (2.33) in the above relations and get

$$\begin{aligned} T^i &= \frac{1}{\xi_2} \int_0^{\xi_2} g^{*1/2} (c^{ij\alpha\beta} \gamma_{\alpha\beta}^* + 2c^{ij\alpha 3} \gamma_{\alpha 3}^* + c^{ij33} \gamma_{33}^*) g_j^* d\xi \\ &\quad + \frac{1}{\xi_2} \int_0^{\xi_2} g^{*1/2} [(-\frac{\kappa_{\alpha\beta} + \kappa_{\beta\alpha}}{2}) c^{ij\alpha\beta} + \kappa_{3\alpha} c^{ij\alpha 3}] \xi g_j^* d\xi \\ &= \gamma_{kl} \frac{1}{\xi_2} \int_0^{\xi_2} c^{ijkl} g^{*1/2} g_j^* d\xi + \frac{1}{2} (\kappa_{\alpha\beta} + \kappa_{\beta\alpha}) \frac{1}{\xi_2} \int_0^{\xi_2} c^{ij\alpha\beta} g^{*1/2} g_j^* \xi d\xi \end{aligned}$$

$$+ \kappa_{3\alpha} \frac{1}{\xi_2} \int_0^{\xi_2} c^{ija3} g^{*1/2} g_j^* \xi d\xi \quad (3.3)$$

We calculate each integral separately, noting that

$$g^{*1/2} g_\gamma^* = g^{1/2} [g_\gamma + \xi \left(\frac{\Delta}{2g} g_\gamma + d_{,\gamma} \right) + \frac{\Delta}{2g} d_{,\gamma} \xi^2] \quad (3.4)$$

$$g^{*1/2} g_3^* = g^{1/2} (g_3 + \frac{\Delta}{2g} \xi g_3) = g^{1/2} (1 + \frac{\Delta}{2g} \xi) g_3 \quad (3.5)$$

The first integral in (3.3) is

$$\frac{1}{\xi_2} \int_0^{\xi_2} c^{ijk} g^{*1/2} g_j^* d\xi = \frac{1}{\xi_2} \int_0^{\xi_2} (c^{i\gamma k} g^{*1/2} g_\gamma^* + c^{i3k} g^{*1/2} g_3^*) d\xi \quad (3.6)$$

The first term of (3.6) from (3.4) is equal to

$$g^{1/2} g_\gamma \frac{1}{\xi_2} \int_0^{\xi_2} c^{i\gamma k} d\xi + g^{1/2} \left(\frac{\Delta}{2g} g_\gamma + d_{,\gamma} \right) \frac{1}{\xi_2} \int_0^{\xi_2} \xi c^{i\gamma k} d\xi + \frac{\Delta}{2g^{1/2}} \frac{d_{,\gamma}}{\xi_2} \int_0^{\xi_2} \xi^2 c^{i\gamma k} d\xi \quad (3.7)$$

and its second term can be written as, from (3.5),

$$g^{1/2} g_3 \left(\frac{1}{\xi_2} \int_0^{\xi_2} c^{i3k} d\xi + \frac{\Delta}{2g} \frac{1}{\xi_2} \int_0^{\xi_2} \xi c^{i3k} d\xi \right) \quad (3.8)$$

Combining (3.7) and (3.8) we rewrite (3.6) as

$$\begin{aligned} g^{1/2} g_j \frac{1}{\xi_2} \int_0^{\xi_2} c^{ijk} d\xi + \frac{\Delta}{2g^{1/2}} g_j \frac{1}{\xi_2} \int_0^{\xi_2} \xi c^{ijk} d\xi + g^{1/2} d_{,\gamma} \frac{1}{\xi_2} \int_0^{\xi_2} \xi c^{i\gamma k} d\xi \\ + \frac{\Delta}{2g^{1/2}} d_{,\gamma} \frac{1}{\xi_2} \int_0^{\xi_2} \xi^2 c^{i\gamma k} d\xi \end{aligned} \quad (3.9)$$

The second integral in (3.3) is

$$\frac{1}{\xi_2} \int_0^{\xi_2} c^{ij\alpha\beta} g^{*1/2} g_j^* \xi d\xi = \frac{1}{\xi_2} \int_0^{\xi_2} (\xi c^{i\gamma\alpha\beta} g^{*1/2} g_\gamma^* + \xi c^{i3\alpha\beta} g^{*1/2} g_3^*) d\xi \quad (3.10)$$

The first term of (3.10) from (3.4) is

$$g^{1/2} g_\gamma \frac{1}{\xi_2} \int_0^{\xi_2} \xi c^{i\gamma\alpha\beta} d\xi + g^{1/2} \left(\frac{\Delta}{2g} g_\gamma + d_{,\gamma} \right) \frac{1}{\xi_2} \int_0^{\xi_2} \xi^2 c^{i\gamma\alpha\beta} d\xi \\ + \frac{\Delta}{2g^{1/2}} \frac{d_\gamma}{\xi_2} \int_0^{\xi_2} \xi^3 c^{i\gamma\alpha\beta} d\xi \quad (3.11)$$

and its second term from (3.5) is

$$g^{1/2} g_3 \left(\frac{1}{\xi_2} \int_0^{\xi_2} \xi c^{i3\alpha\beta} d\xi + \frac{\Delta}{2g} \frac{1}{\xi_2} \int_0^{\xi_2} \xi^2 c^{i3\alpha\beta} d\xi \right) \quad (3.12)$$

Combining (3.11) and (3.12) we rewrite (3.10) as

$$g^{1/2} g_j \frac{1}{\xi_2} \int_0^{\xi_2} \xi c^{ij\alpha\beta} d\xi + \frac{\Delta}{2g^{1/2}} g_j \frac{1}{\xi_2} \int_0^{\xi_2} \xi^2 c^{ij\alpha\beta} d\xi + g^{1/2} d_{,\gamma} \frac{1}{\xi_2} \int_0^{\xi_2} \xi^2 c^{i\gamma\alpha\beta} d\xi \\ + \frac{\Delta}{2g^{1/2}} d_{,\gamma} \frac{1}{\xi_2} \int_0^{\xi_2} \xi^3 c^{i\gamma\alpha\beta} d\xi \quad (3.13)$$

The third integral in (3.3) is

$$\frac{1}{\xi_2} \int_0^{\xi_2} \xi c^{ij\alpha 3} g^{*1/2} g_j^* d\xi = g^{1/2} g_j \frac{1}{\xi_2} \int_0^{\xi_2} \xi c^{ij\alpha 3} d\xi + \frac{\Delta}{2g^{1/2}} g_j \frac{1}{\xi_2} \int_0^{\xi_2} \xi^2 c^{ij\alpha 3} d\xi \\ + g^{1/2} d_{,\gamma} \frac{1}{\xi_2} \int_0^{\xi_2} \xi^2 c^{i\gamma\alpha 3} d\xi + \frac{\Delta}{2g^{1/2}} d_{,\gamma} \frac{1}{\xi_2} \int_0^{\xi_2} \xi^3 c^{i\gamma\alpha 3} d\xi \quad (3.14)$$

The last result was written by noting the development in (2.13). The results in (3.9), (3.13) and (3.14) can further be simplified by recalling (2.9)₃, namely $d_{,\gamma} = \lambda_{\gamma j}^j g_j$ and using the following definitions and results.

Define

$$m = \xi_1/\xi_2 < 1 \quad (3.15)$$

$$a = \begin{cases} a_1 & 0 < \xi < \xi_1 \\ a_2 & \xi_1 < \xi < \xi_2 \end{cases} \quad (3.16)$$

Then

$$\frac{1}{\xi_2} \int_0^{\xi_2} \xi^k a \, d\xi = \frac{\xi_1^k}{k+1} [ma_1 + (\frac{1}{m^k} - m)a_2] \quad k \neq -1 \quad (3.17)$$

Now (3.9) is equal to

$$\begin{aligned} & g^{1/2} g_j \{ mc_1^{ijk'l} + (1-m)c_2^{ijk'l} + \frac{\xi_1 \Delta}{4g} [mc_1^{ijk'l} + (\frac{1}{m} - m)c_2^{ijk'l}] \\ & + \lambda_\gamma^j \frac{\xi_1}{2} [mc_1^{i\gamma k'l} + (\frac{1}{m} - m)c_2^{i\gamma k'l}] + \frac{\xi_1^2 \Delta}{6g} \lambda_\gamma^j [mc_1^{i\gamma k'l} + (\frac{1}{m^2} - m)c_2^{i\gamma k'l}] \} \\ & = g^{1/2} g_j \{ (1 + \frac{\xi_1 \Delta}{4g}) mc_1^{ijk'l} + \frac{\xi_1}{2} \lambda_\gamma^j (1 + \frac{\xi_1 \Delta}{3g}) mc_1^{i\gamma k'l} \\ & + (1-m)(1 + \frac{1+m}{4m} \frac{\xi_1 \Delta}{g}) c_2^{ijk'l} + \frac{\xi_1(1-m)}{2m} \lambda_\gamma^j (1 + m \\ & + \frac{\xi_1 \Delta}{3g} \frac{1+m+m^2}{m}) c_2^{i\gamma k'l} \} \end{aligned} \quad (3.18)$$

(3.13) can also be written as

$$\begin{aligned} & g^{1/2} g_j \{ [mc_1^{ij\alpha\beta} + (\frac{1}{m} - m)c_2^{ij\alpha\beta}] \frac{\xi_1}{2} + \frac{\Delta}{2g} \frac{\xi_1^2}{3} [mc_1^{ij\alpha\beta} + (\frac{1}{m^2} - m)c_2^{ij\alpha\beta}] \\ & + \lambda_\gamma^j \frac{\xi_1^2}{3} [mc_1^{i\gamma\alpha\beta} + (\frac{1}{m^2} - m)c_2^{i\gamma\alpha\beta}] + \frac{\Delta}{2g} \lambda_\gamma^j \frac{\xi_1^3}{4} [mc_1^{i\gamma\alpha\beta} + (\frac{1}{m^3} - m)c_2^{i\gamma\alpha\beta}] \} \end{aligned}$$

$$\begin{aligned}
 &= g^{1/2} g_j \left\{ \left(\frac{\xi_1}{2} + \frac{\xi_1^2 \Delta}{6g} \right) m c_1^{ija\beta} + \frac{\xi_1^2}{3} \lambda_j^j \left(1 + \frac{3}{8} \frac{\xi_1 \Delta}{g} \right) m c_1^{i\gamma\alpha\beta} \right. \\
 &\quad + \frac{\xi_1(1-m)}{2m} \left(1 + m + \frac{\xi_1 \Delta}{3g} \frac{1+m+m^2}{m} \right) c_2^{ija\beta} + \frac{\xi_1^2 \lambda_j^j}{3} \left[\frac{1}{m^2} - m \right. \\
 &\quad \left. \left. + \frac{3}{8} \frac{\xi_1 \Delta}{g} \left(\frac{1}{m^3} - m \right) \right] c_2^{ija\beta} \right\} \quad (3.19)
 \end{aligned}$$

The expression for (3.14) is exactly the same as (3.19) except that (α, β) in (3.19) should be replaced by $(\alpha, 3)$. These results should be incorporated in (3.3) to find an expression for T^i . Due to the presence of the factor $g^{1/2} g_j$ in all these expressions and also the equality (2.66), we can find the constitutive relation for τ^{ij} . However before doing so we exploit the symmetry of c^{ijkl} to further simplify (3.3). Since $c^{ija\beta} = c^{ij\beta\alpha}$ we can write

$$c^{ija\beta} \kappa_{\beta\alpha} = c^{ij\beta\alpha} \kappa_{\beta\alpha} = c^{ija\beta} \kappa_{\alpha\beta} \quad (3.20)$$

Therefore,

$$c^{ija\beta} \kappa_{\beta\alpha} = \frac{1}{2} c^{ija\beta} (\kappa_{\alpha\beta} + \kappa_{\beta\alpha}) \quad (3.21)$$

In view of (3.21), now we write (3.3) as

$$T^i = \gamma_{kl} \frac{1}{\xi_2} \int_0^{\xi_2} c^{ijkl} g^{*1/2} g_j^* d\xi + \kappa_{\alpha\beta} \frac{1}{\xi_2} \int_0^{\xi_2} c^{ij\alpha\beta} g^{*1/2} g_j^* \xi d\xi \quad (3.22)$$

With the explanation presented above, now we write the constitutive relation for τ^{ij} :

$$\begin{aligned}
 \tau^{ij} = & \left\{ \left(1 + \frac{\xi_1 \Delta}{4g} \right) m c_1^{ijkl} + \frac{\xi_1}{2} \lambda_j^j \left(1 + \frac{\xi_1 \Delta}{3g} \right) m c_1^{i\gamma k l} \right. \\
 & + (1-m) \left(1 + \frac{1+m}{4m} \frac{\xi_1 \Delta}{g} \right) c_2^{ijkl} + \frac{\xi_1(1-m)}{2m} \lambda_j^j \left(1 + m + \frac{\xi_1 \Delta}{3g} \frac{1+m+m^2}{m} \right) c_2^{i\gamma k l} \Big\} \gamma_{kl} \\
 & + \left\{ \left(1 + \frac{\xi_1 \Delta}{3g} \right) \frac{m \xi_1}{2} c_1^{ij\alpha\beta} + \frac{\xi_1^2}{3} \lambda_j^j \left(1 + \frac{3}{8} \frac{\xi_1 \Delta}{g} \right) m c_1^{i\gamma\alpha\beta} \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\xi_1(1-m)}{2m} (1+m + \frac{\xi_1\Delta}{3g} \frac{1+m+m^2}{m}) c_2^{ijk\alpha} + \frac{\xi_1^2 \lambda_j^j}{3} [\frac{1}{m^2} - m \\
 & + \frac{3}{8} \frac{\xi_1\Delta}{g} (\frac{1}{m^3} - m)] c_2^{ijk\alpha} \kappa_{k\alpha}
 \end{aligned} \quad (3.23)$$

If we further restrict ourselves to small deformations of a composite with initially flat plies, equation (3.23) can be simplified to

$$\tau^{ij} = \{mc_1^{ijk} + (1-m)c_2^{ijk}\} \gamma_{kl} + \frac{\xi_1}{2} \{mc_1^{ijk\alpha} + \frac{1-m^2}{m} c_2^{ijk\alpha}\} \kappa_{k\alpha} \quad (3.24)$$

Of course in the above equations no distinction should be made between covariant and contravariant components of tensors. Using (2.114), equation (3.24) can be written in terms of displacement vector u and director displacement vector δ , hence

$$\begin{aligned}
 \tau_{ij} &= \{mc_{ijk}^{(1)} + (1-m)c_{ijk}^{(2)}\} \gamma_{kl} + \frac{\xi_1}{2} \{mc_{ijk\alpha}^{(1)} + \frac{1-m^2}{m} c_{ijk\alpha}^{(2)}\} \kappa_{k\alpha} \\
 &= \{mc_{ij\alpha\beta}^{(1)} + (1-m)c_{ij\alpha\beta}^{(2)}\} \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha}) + 2\{mc_{ij\alpha 3}^{(1)} + (1-m)c_{ij\alpha 3}^{(2)}\} (u_{3,\alpha} + \delta_{\alpha})/2 \\
 &\quad + \{mc_{ij33}^{(1)} + (1-m)c_{ij33}^{(2)}\} \delta_3 + \frac{\xi_1}{2} \{mc_{ijk\alpha}^{(1)} + \frac{1-m^2}{m} c_{ijk\alpha}^{(2)}\} \delta_{l,\alpha}
 \end{aligned} \quad (3.25)$$

Using the symmetry of c_{ijkl} 's (3.25) can be written as

$$\begin{aligned}
 \tau_{ij} &= \{mc_{ij\alpha\beta}^{(1)} + (1-m)c_{ij\alpha\beta}^{(2)}\} u_{\alpha,\beta} + \{mc_{ij\alpha 3}^{(1)} + (1-m)c_{ij\alpha 3}^{(2)}\} u_{3,\alpha} \\
 &\quad + \{mc_{ij\alpha 3}^{(1)} + (1-m)c_{ij\alpha 3}^{(2)}\} \delta_{\alpha} + \{mc_{ij33}^{(1)} + (1-m)c_{ij33}^{(2)}\} \delta_3 \\
 &\quad + \frac{\xi_1}{2} \{mc_{ijk\alpha}^{(1)} + \frac{(1-m^2)}{m} c_{ijk\alpha}^{(2)}\} \delta_{l,\alpha} \\
 &= \{mc_{ijk\alpha}^{(1)} + (1-m)c_{ijk\alpha}^{(2)}\} u_{k,\alpha} + \{mc_{ijk3}^{(1)} + (1-m)c_{ijk3}^{(2)}\} \delta_k
 \end{aligned}$$

$$+ \frac{\xi_1}{2} \{ mc_{ijk\alpha}^{(1)} + \frac{(1-m^2)}{m} c_{ijk\alpha}^{(2)} \} \delta_{i,\alpha} \quad (3.26)$$

The same steps can be followed to calculate S^α and we record the procedure here KS

$$\begin{aligned} S^\alpha &= \frac{1}{\xi_2} \int_0^{\xi_2} \xi T^{\alpha} d\xi = \frac{1}{\xi_2} \int_0^{\xi_2} \xi g^{*1/2} \tau^{\alpha j} g_j^* d\xi \\ &= \frac{1}{\xi_2} \int_0^{\xi_2} \xi g^{*1/2} c^{\alpha j k l} \gamma_{kl}^* g_j^* d\xi = \frac{1}{\xi_2} \int_0^{\xi_2} \xi g^{*1/2} (c^{\alpha j \lambda \beta} \gamma_{\lambda \beta}^* + 2c^{\alpha j \lambda 3} \gamma_{\lambda 3}^* \\ &\quad + c^{\alpha j 33} \gamma_{33}^*) g_j^* d\xi = \gamma_{kl} \frac{1}{\xi_2} \int_0^{\xi_2} \xi g^{*1/2} c^{\alpha j k l} g_j^* d\xi + \kappa_{\beta} \frac{1}{\xi_2} \int_0^{\xi_2} \xi^2 g^{*1/2} c^{\alpha j \lambda \beta} g_j^* d\xi \end{aligned} \quad (3.27)$$

This is basically the same result as (3.22) except that i has been replaced by α and the integrand has been multiplied by ξ . The first integral can be reduced to the following by referring to (3.18)

$$\begin{aligned} g^{1/2} g_j \{ \frac{\xi_1}{2} [mc_1^{\alpha j k l} + (\frac{1}{m} - m) c_2^{\alpha j k l}] + \frac{\xi_1^2 \Delta}{6g} [mc_1^{\alpha j k l} + (\frac{1}{m^2} - m) c_2^{\alpha j k l}] \\ \lambda_{\gamma}^j \frac{\xi_1^2}{3} [mc_1^{\alpha \gamma k l} + (\frac{1}{m^2} - m) c_2^{\alpha \gamma k l}] + \frac{\xi_1^3 \Delta}{8g} \lambda_{\gamma}^j [mc_1^{\alpha \gamma k l} + (\frac{1}{m^3} - m) c_2^{\alpha \gamma k l}] \} \end{aligned} \quad (3.28)$$

Similarly the second integral is simplified and by reference to (3.19) the result is

$$\begin{aligned} g^{1/2} g_j \{ \frac{\xi_1^2}{3} [mc_1^{\alpha j \beta} + (\frac{1}{m^2} - m) c_2^{\alpha j \beta}] + \frac{\xi_1^3 \Delta}{8g} [mc_1^{\alpha j \beta} + (\frac{1}{m^3} - m) c_2^{\alpha j \beta}] \\ + \lambda_{\gamma}^j \frac{\xi_1^3}{4} [mc_1^{\alpha \gamma \beta} + (\frac{1}{m^3} - m) c_2^{\alpha \gamma \beta}] + \lambda_{\gamma}^j \frac{\xi_1^4 \Delta}{10g} [mc_1^{\alpha \gamma \beta} + (\frac{1}{m^4} - m) c_2^{\alpha \gamma \beta}] \} \\ = \xi_1^2 g^{1/2} g_j \{ (\frac{m}{3} + \frac{m\Delta\xi_1}{8g}) c_1^{\alpha j \beta} + \lambda_{\gamma}^j \xi_1 (\frac{1}{4} + \frac{\xi_1 \Delta}{10g}) mc_1^{\alpha \gamma \beta} \\ + [\frac{1}{3} (\frac{1-m^3}{m^2}) + \frac{\xi_1 \Delta}{8g} \frac{1-m^4}{m^3}] c_2^{\alpha j \beta} + \lambda_{\gamma}^j \xi_1 (\frac{1-m^4}{4m^3} + \frac{\xi_1 \Delta}{10g} \frac{1-m^5}{m^4}) c_2^{\alpha \gamma \beta} \} \end{aligned} \quad (3.29)$$

Again it is seen that because of the factor $g^{1/2} g_j$ and the relation $S^\alpha = S^{\alpha j} g^{1/2} g_j$ we can readily write the constitutive relation for $S^{\alpha j}$, the result is

$$\begin{aligned}
 S^{\alpha j} = & \xi_1 \left\{ \frac{1}{2} (mc_1^{\alpha j k l}) \left(1 + \frac{\xi_1 \Delta}{3g} \right) + \xi_1 \lambda_j^k \left(\frac{1}{3} + \frac{\xi_1 \Delta}{8g} \right) mc_1^{\alpha \gamma k l} \right. \\
 & + \frac{1}{2} \left(\frac{1-m^2}{m} + \frac{1-m^3}{m^2} \frac{\xi_1 \Delta}{3g} \right) c_2^{\alpha j k l} + \lambda_j^k \xi_1 \left(\frac{1-m^3}{3m^2} + \frac{1-m^4}{m^3} \frac{\xi_1 \Delta}{8g} \right) c_2^{\alpha \gamma k l} \gamma_{kl} \\
 & + \xi_1^2 \left\{ \left(\frac{1}{3} + \frac{\xi_1 \Delta}{8g} \right) mc_1^{\alpha j \beta} + \xi_1 \lambda_j^k \left(\frac{1}{4} + \frac{\xi_1 \Delta}{10g} \right) mc_1^{\alpha \gamma \beta} + \left(\frac{1-m^3}{3m^2} + \frac{\xi_1 \Delta}{8g} \frac{1-m^4}{m^3} \right) c_2^{\alpha j \beta} \right. \\
 & \left. \left. + \xi_1 \lambda_j^k \left(\frac{1-m^4}{4m^3} + \frac{\xi_1 \Delta}{10g} \frac{1-m^5}{m^4} \right) c_2^{\alpha \gamma \beta} \right\} \kappa_{\beta} \right\} \quad (3.30)
 \end{aligned}$$

This is the general constitutive relation for $S^{\alpha j}$ in linear elasticity. If, as before, we confine ourselves to small deformations of a composite with initially flat plies (3.30) can be simplified to

$$S^{\alpha j} = \frac{1}{2} \xi_1 (mc_1^{\alpha j k l} + \frac{1-m^2}{m} c_2^{\alpha j k l}) \gamma_{kl} + \frac{1}{3} \xi_1^2 (mc_1^{\alpha j \beta} + \frac{1-m^3}{m^2} c_2^{\alpha j \beta}) \kappa_{\beta} \quad (3.31)$$

with no distinction between covariant and contravariant tensors. Once written in terms of displacement vector and director displacement and simplified as done in obtaining (3.26) we get

$$\begin{aligned}
 S^{\alpha j} = & \frac{1}{2} \xi_1 (mc_1^{\alpha j \beta k} + \frac{1-m^2}{m} c_2^{\alpha j \beta k}) u_{k, \beta} + \frac{1}{2} \xi_1 (mc_1^{\alpha j k 3} + \frac{1-m^2}{m} c_2^{\alpha j k 3}) \delta_k \\
 & + \frac{1}{3} \xi_1^2 (mc_1^{\alpha j \beta} + \frac{1-m^3}{m^2} c_2^{\alpha j \beta}) \delta_{L \beta} \quad (3.32)
 \end{aligned}$$

As the results of this section indicate, even in the simplest cases of small deformations of an initially flat composite (composites with flat plies) higher gradients of displacement vector become significant and they appear in the constitutive relations for *composite stress* and *composite stress couple*. As defined by (3.15) m is of the order of unity, ξ_1 and ξ_2 are usually small lengths; however their products with components of c_{ijkl} and even the product of their

higher powers with elastic constants may be indeed significant quantities, in which case we get contributions to τ_{ij} and $S_{\alpha j}$. In the trivial case $m = 1$, $\xi_1 = \xi_2 \rightarrow 0$ we get $\tau_{ij} = c_{ijk}/\eta_{kb}$, $S_{\alpha j} \equiv 0$ and the equations of linear elasticity are recovered.

4.0 COMPLETE THEORY FOR LINEAR ELASTIC COMPOSITE LAMINATES

The results of sections (2) and (3) are combined to obtain the complete equations for a linear elastic composite laminate. However, before doing so we should derive appropriate expression for ρ_o , z^1 and z^2 . As before, we assume that the representative micro-structure is composed of two homogeneous layers with respective densities ρ_1 and ρ_2 in the reference configuration (ρ_1 and ρ_2 are constants). Recalling equations (2.45) and (2.46) we write

$$\rho g^{1/2} = \rho_o G^{1/2} = \frac{1}{\xi_2} \int_0^{\xi_2} \rho_o^* G^{*1/2} d\xi \quad (4.1)$$

Now by (2.13) we have

$$G^{*1/2} = G^{1/2} \left(1 + \frac{\xi \Delta}{2G} \right) \quad (4.2)^*$$

where Δ is understood to be the sum of two determinants similar to those expressed in (2.14) except for substituting g_{ij} by G_{ij} . We have

$$\rho_o^* = \begin{cases} \rho_1 & 0 < \xi < \xi_1 \\ \rho_2 & \xi_1 < \xi < \xi_2 \end{cases} \quad (4.3)$$

Substituting (4.2) and (4.3) in (4.1) and using (3.7) we set

$$\rho_o = \left(1 + \frac{\xi_1 \Delta}{4G} \right) m \rho_1 + \left(1 + \frac{1+m}{4mG} \xi_1 \Delta \right) (1-m) \rho_2 \quad (4.4)$$

Of course the *composite mass density* ρ is related to ρ_o through the equation (2.122).

We proceed similarly to calculate z^1 and z^2 using their definitions in (2.42)

$$\rho g^{1/2} z^1 = \rho_o G^{1/2} z^1 = \frac{1}{\xi_2} \int_0^{\xi_2} \xi \rho_o^* G^{*1/2} d\xi \quad (4.5)$$

* Here again the result has been linearized in terms of ξ .

$$\rho g^{1/2} z^2 = \rho_o G^{1/2} z^2 = \frac{1}{\xi_2} \int_0^{\xi_2} \xi^2 \rho_o^* G^{*1/2} d\xi \quad (4.6)$$

After substituting from (4.2) and (4.3) in (4.5) and (4.6) and using (3.17) we get

$$\rho_o z^1 = \frac{\xi_1}{2} \left[\left(1 + \frac{\xi_1 \Delta}{3G}\right) m \rho_1 + \left(\frac{1-m^2}{m} + \frac{\xi_1 \Delta}{3G} \frac{1-m^3}{m^2}\right) \rho_2 \right] \quad (4.7)$$

$$\rho_o z^2 = \frac{\xi_1^2}{3} \left[\left(1 + \frac{3\xi_1 \Delta}{8G}\right) m \rho_1 + \left(\frac{1-m^3}{m^2} + \frac{3\xi_1 \Delta}{8G} \frac{1-m^4}{m^3}\right) \rho_2 \right] \quad (4.8)$$

For a composite with initially flat plies (4.4), (4.7) and (4.8) are reduced respectively to

$$\rho_o = m \rho_1 + (1-m) \rho_2 \quad (4.9)$$

$$\rho_o z^1 = \frac{\xi_1}{2} \left(m \rho_1 + \frac{1-m^2}{m} \rho_2 \right) \quad (4.10)$$

$$\rho_o z^2 = \frac{\xi_1^2}{3} \left(m \rho_1 + \frac{1-m^3}{m^2} \rho_2 \right) \quad (4.11)$$

To formulate the complete theory it is also worthwhile to derive a relation between the director displacement δ and the gradient of displacement vector u in the θ^3 -direction. In order to derive such a relation we enforce the continuity of the position vectors p^* and P^* between two adjacent micro-structures. Recalling (2.1) and (2.2), we have the following relations for the k^{th} micro-structure:

$$p^*(\theta^\alpha, \theta^{3(k)}, \xi, t) = r(\theta^\alpha, \theta^{3(k)}, t) + \xi d(\theta^\alpha, \theta^{3(k)}, t) \quad (4.12)$$

$$P^*(\theta^\alpha, \theta^{3(k)}, \xi) = R(\theta^\alpha, \theta^{3(k)}) + \xi D(\theta^\alpha, \theta^{3(k)}) \quad (4.13)$$

Now in order that position vectors p^* and P^* be continuous on the common surface between k^{th} and $(k+1)^{\text{st}}$ micro-structures we should have

$$P^*(\theta^\alpha, \theta^{3(k)}, \xi_2) = P^*(\theta^\alpha, \theta^{3(k+1)}, 0) \quad (4.14)$$

$$p^*(\theta^\alpha, \theta^{3(k)}, \xi_2, t) = p^*(\theta^\alpha, \theta^{3(k+1)}, 0, t) \quad (4.15)$$

Using (4.12) and (4.13) we can write (4.14) and (4.15) as

$$R(\theta^\alpha, \theta^{3(k+1)}) = R(\theta^\alpha, \theta^{3(k)}) + \xi_2 D(\theta^\alpha, \theta^{3(k)}) \quad (4.16)$$

$$r(\theta^\alpha, \theta^{3(k+1)}, t) = r(\theta^\alpha, \theta^{3(k)}, t) + \xi_2 d(\theta^\alpha, \theta^{3(k)}, t) \quad (4.17)$$

Recalling (2.98) and (2.99) and identifying εu and $\varepsilon \delta$ with u and δ as before we conclude from (4.16) and (4.17) the following

$$u(\theta^\alpha, \theta^{3(k+1)}, t) = u(\theta^\alpha, \theta^{3(k)}, t) + \xi_2 \delta(\theta^\alpha, \theta^{3(k)}, t) \quad (4.18)$$

or

$$\delta(\theta^\alpha, \theta^{3(k)}, t) = \frac{1}{\xi_2} [u(\theta^\alpha, \theta^{3(k+1)}, t) - u(\theta^\alpha, \theta^{3(k)}, t)] \quad (4.19)$$

By smoothing assumptions and approximating the right-hand side of equation (4.19) as the gradient of the displacement vector in the θ^3 direction we have

$$\delta(\theta^\alpha, \theta^3, t) = \frac{\partial u(\theta^\alpha, \theta^3)}{\partial \theta^3} \quad (4.20)$$

In component form we have

$$\delta = \delta^j g_j = (u^j g_j)_{,3} = u^j_{,3} g_j \quad (4.21)$$

or

$$\delta^j = u^j_{,3} \quad , \quad \delta_j = u_{j,3} \quad (4.22)$$

For a composite with initially flat plies the equation (4.22) reduces to

$$\delta_j = u_{j,3} \quad (4.23)$$

With this simplification, equations (2.119) reduce to

$$\gamma_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (4.24)$$

$$\kappa_{j\alpha} = u_{j,\alpha 3} \quad (4.25)$$

Using (4.22), equations (2.121) and (2.125) are also reduced to

$$\rho_0 = \rho \left(\frac{g}{G} \right)^{1/2} = \rho (1 + u^j_{,j}) \quad (4.26)$$

$$\rho = \rho_0 (1 - u^j_{,j}) \quad (4.27)$$

The constitutive relations (3.26) and (3.32) for τ_{ij} and $S_{\alpha j}$ for a flat composite are also further simplified by using (4.23)

$$\tau_{ij} = \{ m c_{ijk}^{(1)} + (1-m) c_{ijk}^{(2)} \} u_{k,l} + \{ m c_{ijk\alpha}^{(1)} + \frac{1-m^2}{m} c_{ijk\alpha}^{(2)} \} \frac{\xi_1}{2} u_{k,\alpha 3} \quad (4.28)$$

and

$$S_{\alpha j} = \frac{1}{2} \xi_1 \{ m c_{ijk}^{(1)} + \frac{1-m^2}{m} c_{ijk}^{(2)} \} u_{k,l} + \frac{1}{3} \xi_1^2 \{ m c_{\alpha jk\beta} + \frac{1-m^3}{m^2} c_{\alpha jk\beta}^{(2)} \} u_{k,\beta 3} \quad (4.29)$$

Now we can write the field equations (2.113) and (2.124) in terms of the displacement vector u and its gradients. It should be recalled that the resulting equations are the linearized field equation for small deformations of a composite with initially flat plies. These are the counterpart of the Navier-Cauchy equations in linear elasticity. The appropriate equations for a general composite will be derived in later chapters. Using (4.9)-(4.11), (4.23), (4.28) and (4.29) we write (2.123) and (2.124) as

$$\begin{aligned}
 & \{mc_{\alpha j k l}^{(1)} + (1-m)c_{\alpha j k l}^{(2)}\}u_{k,l\alpha} + \{mc_{\alpha j k \beta}^{(1)} + \frac{1-m^2}{m} c_{\alpha j k \beta}^{(2)}\} \frac{\xi_1}{2} u_{k,\beta 3\alpha} \\
 & + \sigma_{j,3} + [mp_1 + (1-m)\rho_2]b_j = [mp_1 + (1-m)\rho_2]\ddot{u}_j \\
 & + \frac{1}{2} \xi_1(mp_1 + \frac{1-m^2}{m} \rho_2)\ddot{u}_{j,3}
 \end{aligned} \tag{4.30}$$

and

$$\begin{aligned}
 & \frac{1}{2} \xi_1 \{mc_{\alpha j k l}^{(1)} + \frac{1-m^2}{m} c_{\alpha j k l}^{(2)}\}u_{k,l\alpha} + \frac{\xi_1^2}{3} \{mc_{\alpha j k \beta}^{(1)} + \frac{1-m^3}{m^2} c_{\alpha j k \beta}^{(1)}\}u_{k,\beta 3\alpha} \\
 & + \sigma_j - [mc_{3 j k l}^{(1)} + (1-m)c_{3 j k l}^{(2)}]u_{k,l} - \frac{\xi_1}{2} \{mc_{3 j k \alpha}^{(1)} + \frac{1-m^2}{m} c_{3 j k \alpha}^{(2)}\}u_{k,\alpha 3} \\
 & + [mp_1 + (1-m)\rho_o]c_j = \frac{1}{2} \xi_1(mp_1 + \frac{1-m^2}{m} \rho_2)\ddot{u}_j \\
 & + \frac{1}{3} \xi_1^2(mp_1 + \frac{1-m^3}{m^2} \rho_2)\ddot{u}_{j,3}
 \end{aligned} \tag{4.31}$$

At this point we may notice that an ordinary continuum (a single material continuum) can be regarded as the limiting case of a composite laminate when $\xi_1 = \xi_2 \rightarrow 0$. Therefore, we may anticipate to derive the equations of linear elasticity by letting $m = 1$ and $\xi_1 \rightarrow 0$ in equations (4.30) and (4.31). Doing so, equation (4.30) reduces to

$$c_{\alpha j k l}u_{k,l\alpha} + \sigma_{j,3} + \rho b_j = \ddot{\rho}u_j \tag{4.32}$$

where subscript and superscript 1 are dropped because we have only one material. To simplify equation (4.31), first we recall the definition of c in equation (2.44) and notice that by the mean-value theorem, $c \rightarrow 0$ as $\xi_2 \rightarrow 0$, hence

$$\sigma_j - c_{3 j k l}u_{k,l} = 0 \tag{4.33}$$

Substituting for σ_j from (4.33) in (4.32) we get

$$c_{\alpha jkl} u_{k,\alpha} + c_{3 jkl} u_{k,3} + \rho b_j = \rho \ddot{u}_j \quad (4.34)$$

and combining the first and the second terms we get

$$c_{ijk} u_{k,i} + \rho b_j = \rho \ddot{u}_j \quad (4.35)$$

For a completely isotropic continuum

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (4.36)$$

where λ and μ are Lamé constants and (4.35) reduces to:

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ij} + \rho b_i = \rho \ddot{u}_i \quad (4.37)$$

which are the equations of motion for an isotropic media.

For the case of the composite laminate we can also eliminate σ_j between equations (4.30) and (4.31) to obtain the appropriate equations for displacement vector u . First we do this for a static problem with no body force. For such a case we let

$$b = c = \dot{u} = 0 \quad (4.38)$$

in equations (4.30) and (4.31), hence

$$\{m c_{\alpha jkl}^{(1)} + (1-m) c_{\alpha jkl}^{(2)}\} u_{k,\alpha} + \{m c_{\alpha jk\beta}^{(1)} + \frac{1-m^2}{m} c_{\alpha jk\beta}^{(2)}\} \frac{\xi_1}{2} u_{k,\beta 3\alpha} + \sigma_{j,3} = 0$$

and

$$\begin{aligned} & \frac{\xi_1}{2} \{mc_{\alpha j k l}^{(1)} + \frac{1-m^2}{m} c_{\alpha j k l}^{(2)}\} u_{k, l \alpha} + \frac{\xi_1^2}{3} \{mc_{\alpha j k \beta}^{(1)} + \frac{1-m^3}{m^2} c_{\alpha j k \beta}^{(2)}\} u_{k, \beta 3 \alpha} \\ & + \sigma_j - \{mc_{3 j k l}^{(1)} + (1-m)c_{3 j k l}^{(2)}\} u_{k, l} - \frac{\xi_1}{2} \{mc_{3 j k \alpha}^{(1)} + \frac{1-m^2}{m} c_{3 j k \alpha}^{(2)}\} u_{k, \alpha 3} = 0 \end{aligned}$$

Eliminating σ_j between these equations, we get

$$\begin{aligned} & \{mc_{\alpha j k l}^{(1)} + (1-m)c_{\alpha j k l}^{(2)}\} u_{k, l \alpha} + \{mc_{\alpha j k \beta}^{(1)} + \frac{1-m^2}{m} c_{\alpha j k \beta}^{(2)}\} \frac{\xi_1}{2} u_{k, \beta 3 \alpha} \\ & + \{mc_{3 j k l}^{(1)} + (1-m)c_{3 j k l}^{(2)}\} u_{k, l 3} + \frac{\xi_1}{2} \{mc_{3 j k \alpha}^{(1)} + \frac{1-m^2}{m} c_{3 j k \alpha}^{(2)}\} u_{k, \alpha 3 3} \\ & - \frac{\xi_1}{2} \{mc_{\alpha j k l}^{(1)} + \frac{1-m^2}{m} c_{\alpha j k l}^{(2)}\} u_{k, l \alpha 3} - \frac{\xi_1^2}{3} \{mc_{\alpha j k \beta}^{(1)} + \frac{1-m^3}{m^2} c_{\alpha j k \beta}^{(2)}\} u_{k, \beta \alpha 3 3} = 0 \quad (4.39) \end{aligned}$$

By combining the first and third and also the second and fourth terms of the equation (4.39), we get

$$\begin{aligned} & \{mc_{ij k l}^{(1)} + (1-m)c_{ij k l}^{(2)}\} u_{k, l i} + \frac{\xi_1}{2} \{mc_{ij k \alpha}^{(1)} + \frac{1-m^2}{m} c_{ij k \alpha}^{(2)}\} u_{k, i \alpha 3} \\ & - \frac{\xi_1}{2} \{mc_{\alpha j k l}^{(1)} + \frac{1-m^2}{m} c_{\alpha j k l}^{(2)}\} u_{k, l \alpha 3} - \frac{\xi_1^2}{3} \{mc_{\alpha j k \beta}^{(1)} + \frac{1-m^3}{m^3} c_{\alpha j k \beta}^{(2)}\} u_{k, \beta \alpha 3 3} = 0 \quad (4.40) \end{aligned}$$

This is a fourth order partial differential equation for displacement vector u . Now we apply this equation to a composite laminate whose micro-structure is composed of two isotropic layers. For such a case we can write

$$c_{ij k l}^{(1)} = \lambda_1 \delta_{ij} \delta_{kl} + \mu_1 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (4.41)$$

$$c_{ij k l}^{(2)} = \lambda_2 \delta_{ij} \delta_{kl} + \mu_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (4.42)$$

where λ_i and μ_i ($i = 1, 2$) are Lamé's constants for the respective layers. Introducing equations (4.41) and (4.42) in (4.40) we get the following for each term:

$$\begin{aligned}
 \{mc_{ijkl}^{(1)} + (1-m)c_{ijkl}^{(1)}\}u_{k,li} &= m\{\lambda_1 u_{k,kj} + \mu_1(u_{k,jk} + u_{j,ll})\} \\
 &+ (1-m)\{\lambda_2 u_{k,kj} + \mu_2(u_{k,jk} + u_{j,ll})\} \\
 &= \{m(\lambda_1 + \mu_1) + (1-m)(\lambda_2 + \mu_2)\}u_{k,kj} + \{m\mu_1 + (1-m)\mu_2\}u_{j,ll}
 \end{aligned} \tag{4.43}$$

$$\begin{aligned}
 \{mc_{ijk\alpha}^{(1)} + \frac{(1-m^2)}{m} c_{ijk\alpha}^{(2)}\}u_{k,i\alpha 3} &= m\{\lambda_1 \delta_{ij} \delta_{k\alpha} + \mu_1(\delta_{ik} \delta_{j\alpha} + \delta_{i\alpha} \delta_{jk})\}u_{k,i\alpha 3} \\
 &+ \frac{(1-m^2)}{m} \{\lambda_2 \delta_{ij} \delta_{k\alpha} + \mu_2(\delta_{ik} \delta_{j\alpha} + \delta_{i\alpha} \delta_{jk})\}u_{k,i\alpha 3} \\
 &= m\{\lambda_1 \delta_{ij} \delta_{\beta\alpha} u_{\beta,i\alpha 3}\} + m\mu_1\{\delta_{ik} \delta_{j\alpha} u_{k,i\alpha 3} + \delta_{\beta\alpha} \delta_{jk} u_{k,\beta\alpha 3}\} \\
 &+ \frac{(1-m^2)}{m} \lambda_2(\delta_{ij} \delta_{\beta\alpha} u_{\beta,i\alpha 3}) + \frac{1-m^2}{m} \mu_2(\delta_{ik} \delta_{j\alpha} u_{k,i\alpha 3} + \delta_{\beta\alpha} \delta_{jk} u_{k,\beta\alpha 3}) \\
 &= \{m\lambda_1 + \frac{(1-m^2)}{m} \lambda_2\}u_{\alpha,j\alpha 3} + \{m\mu_1 + \frac{(1-m^2)}{m} \mu_2\}\delta_{j\alpha} u_{k,k\alpha 3} \\
 &+ \{m\mu_1 + \frac{(1-m^2)}{m} \mu_2\}u_{j,\alpha\alpha 3}
 \end{aligned} \tag{4.44}$$

$$\begin{aligned}
 \{mc_{\alpha jkl}^{(1)} + \frac{(1-m^2)}{m} c_{\alpha jkl}^{(2)}\}u_{k,l\alpha 3} &= m\{\lambda_1 \delta_{\alpha j} \delta_{kl} + \mu_1(\delta_{\alpha k} \delta_{jl} + \delta_{\alpha l} \delta_{jk})\}u_{k,l\alpha 3} \\
 &+ \frac{(1-m^2)}{m} \{\lambda_2 \delta_{\alpha j} \delta_{kl} + \mu_2(\delta_{\alpha k} \delta_{jl} + \delta_{\alpha l} \delta_{jk})\}u_{k,l\alpha 3} \\
 &= m\lambda_1(\delta_{\alpha j} u_{k,k\alpha 3}) + m\mu_1(u_{\alpha,j\alpha 3} + u_{j,\alpha\alpha 3}) \\
 &+ \frac{(1-m^2)}{m} \lambda_2(\delta_{\alpha j} u_{k,k\alpha 3}) + \frac{(1-m^2)}{m} \mu_2(u_{\alpha,j\alpha 3} + u_{j,\alpha\alpha 3}) \\
 &= (m\lambda_1 + \frac{(1-m^2)}{m} \lambda_2)\delta_{\alpha j} u_{k,k\alpha 3} + (m\mu_1 + \frac{(1-m^2)}{m} \mu_2)(u_{\alpha,j\alpha 3} + u_{j,\alpha\alpha 3})
 \end{aligned} \tag{4.45}$$

$$\begin{aligned}
 (mc_{\alpha j k \beta}^{(1)} + \frac{1-m^3}{m^2} c_{\alpha j k \beta}^{(2)}) u_{k, \beta \alpha 33} &= m(\lambda_1 \delta_{\alpha j} \delta_{k \beta} + \mu_1 (\delta_{k \alpha} \delta_{j \beta} + \delta_{\alpha \beta} \delta_{j k})) u_{k, \beta \alpha 33} \\
 &+ \frac{1-m^3}{m^2} \{ \lambda_2 \delta_{\alpha j} \delta_{k \beta} + \mu_2 (\delta_{k \alpha} \delta_{j \beta} + \delta_{\alpha \beta} \delta_{j k}) \} u_{k, \beta \alpha 33} \\
 &= m \lambda_1 \delta_{\alpha j} u_{\beta, \beta \alpha 33} + m \mu_1 (\delta_{j \beta} u_{\alpha, \beta \alpha 33} + u_{j, \alpha \alpha 33}) \\
 &+ \frac{1-m^3}{m^2} \lambda_2 \delta_{\alpha j} u_{\beta, \beta \alpha 33} + \frac{1-m^3}{m^2} \mu_2 (\delta_{j \beta} u_{\alpha, \beta \alpha 33} + u_{j, \alpha \alpha 33}) \\
 &= (m \lambda_1 + \frac{1-m^3}{m^2} \lambda_2) \delta_{\alpha j} u_{\beta, \beta \alpha 33} + (m \mu_1 + \frac{1-m^3}{m^2} \mu_2) (\delta_{j \beta} u_{\alpha, \beta \alpha 33} + u_{j, \alpha \alpha 33}) \quad (4.46)
 \end{aligned}$$

Substituting (4.43)-(4.46) in (4.40) we obtain

$$\begin{aligned}
 \frac{\xi_1}{2} \{ m \lambda_1 + \frac{(1-m^2)}{m} \lambda_2 \} u_{\alpha, j \alpha 3} + \{ m \mu_1 + \frac{(1-m^2)}{m} \mu_2 \} (\delta_{j \alpha} u_{k, k \alpha 3} + u_{j, \alpha \alpha 3}) \frac{\xi_1}{2} \\
 + \{ m(\lambda_1 + \mu_1) + (1-m)(\lambda_2 + \mu_2) \} u_{k, k j} + \{ m \mu_1 + (1-m) \mu_2 \} u_{j, //} \\
 - \frac{\xi_1}{2} (m \lambda_1 + \frac{(1-m^2)}{m} \lambda_2) \delta_{\alpha j} u_{k, k \alpha 3} - \frac{\xi_1}{2} (m \mu_1 + \frac{(1-m^2)}{m} \mu_2) (u_{\alpha, j \alpha 3} + u_{j, \alpha \alpha 3}) \\
 - \frac{\xi_1^2}{3} (m \lambda_1 + \frac{1-m^3}{m^2} \lambda_2) \delta_{\alpha j} u_{\beta, \beta \alpha 33} \\
 - \frac{\xi_1^2}{3} (m \mu_1 + \frac{1-m^3}{m^2} \mu_2) (\delta_{j \beta} u_{\alpha, \beta \alpha 33} + u_{j, \alpha \alpha 33}) = 0
 \end{aligned}$$

Now we introduce the following definitions in the last equation:

$$\lambda_{12} = m\lambda_1 + (1-m)\lambda_2$$

$$\mu_{12} = m\mu_1 + (1-m)\mu_2$$

$$\bar{\lambda}_{12} = m\lambda_1 + \frac{1-m^2}{m} \lambda_2$$

(4.47)

$$\bar{\mu}_{12} = m\mu_1 + \frac{1-m^2}{m} \mu_2$$

$$\bar{\bar{\lambda}}_{12} = m\lambda_1 + \frac{1-m^3}{m^2} \lambda_2$$

$$\bar{\bar{\mu}}_{12} = m\mu_1 + \frac{1-m^3}{m^2} \mu_2$$

The result will be

$$\begin{aligned} & (\lambda_{12} + \mu_{12})u_{k,kj} + \mu_{12}u_{j,l} + \frac{\xi_1}{2} \{ \bar{\lambda}_{12}u_{\alpha,j\alpha 3} + \bar{\mu}_{12}(\delta_{j\alpha}u_{k,k\alpha 3} + u_{j,\alpha\alpha 3}) \} \\ & - \frac{\xi_1}{2} \bar{\lambda}_{12} \delta_{\alpha j}u_{k,k\alpha 3} - \frac{\xi_1}{2} \bar{\mu}_{12}(u_{\alpha,j\alpha 3} + u_{j,\alpha\alpha 3}) \\ & - \frac{\xi_1^2}{3} \bar{\bar{\lambda}}_{12}\delta_{\alpha j}u_{\beta,\beta\alpha 33} - \frac{\xi_1^2}{3} \bar{\bar{\mu}}_{12}(\delta_{j\beta}u_{\alpha,\beta\alpha 33} + u_{j,\alpha\alpha 33}) = 0 \end{aligned} \quad (4.48)$$

The above equations are counterparts of the classical equations for linear elasto-static problems in the absence of body forces. We need to write these equations in the expanded form.

The result would be three scalar equations as follows:

$$\begin{aligned}
 & (\lambda_{12} + \mu_{12}) \frac{\partial}{\partial \theta^1} \left(\frac{\partial u_1}{\partial \theta^1} + \frac{\partial u_2}{\partial \theta^2} + \frac{\partial u_3}{\partial \theta^3} \right) + \mu_{12} \left(\frac{\partial^2 u_1}{\partial \theta_1^2} + \frac{\partial^2 u_1}{\partial \theta_2^2} + \frac{\partial^2 u_1}{\partial \theta_3^2} \right) \\
 & + \frac{\xi_1}{2} \bar{\lambda}_{12} \frac{\partial^2}{\partial \theta_1 \partial \theta_3} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) + \frac{\xi_1}{2} \bar{\mu}_{12} \frac{\partial^2}{\partial \theta_1 \partial \theta_3} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} + \frac{\partial u_3}{\partial \theta_3} \right) \\
 & + \frac{\xi_1}{2} \bar{\mu}_{12} \frac{\partial}{\partial \theta_3} \left(\frac{\partial^3 u_1}{\partial \theta_1^2} + \frac{\partial^2 u_1}{\partial \theta_2^2} \right) - \frac{\xi_1}{2} \bar{\lambda}_{12} \frac{\partial^2}{\partial \theta_1 \partial \theta_3} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} + \frac{\partial u_3}{\partial \theta_3} \right) \\
 & - \frac{\xi_1}{2} \bar{\mu}_{12} \frac{\partial^2}{\partial \theta_1 \partial \theta_3} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) - \frac{\xi_1}{2} \bar{\mu}_{12} \frac{\partial}{\partial \theta_3} \left(\frac{\partial^2 u_1}{\partial \theta_1^2} + \frac{\partial^2 u_1}{\partial \theta_2^2} \right) \\
 & - \frac{\xi_1^2}{3} \bar{\lambda}_{12} \frac{\partial^3}{\partial \theta_1 \partial \theta_3^2} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) - \frac{\xi_1^2}{3} \bar{\mu}_{12} \frac{\partial^3}{\partial \theta_1 \partial \theta_3^2} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) \\
 & - \frac{\xi_1^2}{3} \bar{\mu}_{12} \frac{\partial^2}{\partial \theta_3^2} \left(\frac{\partial^2 u_1}{\partial \theta_1^2} + \frac{\partial^2 u_1}{\partial \theta_2^2} \right) = 0
 \end{aligned} \tag{4.49}$$

This is the first equation for $j = 1$ which can further be simplified as

$$\begin{aligned}
 & (\lambda_{12} + \mu_{12}) \frac{\partial}{\partial \theta^1} \left(\frac{\partial u_1}{\partial \theta^1} + \frac{\partial u_2}{\partial \theta^2} + \frac{\partial u_3}{\partial \theta^3} \right) + \mu_{12} \left(\frac{\partial^2 u_1}{\partial \theta_1^2} + \frac{\partial^2 u_1}{\partial \theta_2^2} + \frac{\partial^2 u_1}{\partial \theta_3^2} \right) \\
 & + \frac{\xi_1}{2} (\bar{\mu}_{12} - \bar{\lambda}_{12}) \frac{\partial^3 u_3}{\partial \theta_1 \partial \theta_3^2} \\
 & - \frac{\xi_1^2}{3} (\bar{\lambda} + \bar{\mu}_{12}) \frac{\partial^3}{\partial \theta_1 \partial \theta_3^2} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) \\
 & - \frac{\xi_1^2}{3} \bar{\mu}_{12} \frac{\partial^2}{\partial \theta_3^2} \left(\frac{\partial^2 u_1}{\partial \theta_1^2} + \frac{\partial^2 u_1}{\partial \theta_2^2} \right) = 0
 \end{aligned} \tag{4.50}$$

The second equation will be

$$\begin{aligned}
 & (\lambda_{12} + \mu_{12}) \frac{\partial}{\partial \theta_2} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta^2} + \frac{\partial u_3}{\partial \theta_3} \right) + \mu_{12} \left(\frac{\partial^2 u_2}{\partial \theta_1^2} + \frac{\partial^2 u_2}{\partial \theta_2^2} + \frac{\partial^2 u_2}{\partial \theta_3^2} \right) \\
 & + \frac{\xi_1}{2} (\bar{\mu}_{12} - \bar{\lambda}_{12}) \frac{\partial^3 u_3}{\partial \theta_2 \partial \theta_3^2} \\
 & - \frac{\xi_1^2}{3} (\bar{\lambda}_{12} + \bar{\mu}_{12}) \frac{\partial^3}{\partial \theta_2 \partial \theta_3^2} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) \\
 & - \frac{\xi_1^2}{3} \bar{\mu}_{12} \frac{\partial^2}{\partial \theta_3^2} \left(\frac{\partial^2 u_2}{\partial \theta_1^2} + \frac{\partial^2 u_2}{\partial \theta_2^2} \right) = 0
 \end{aligned} \tag{4.51}$$

And the third equation is

$$\begin{aligned}
 & (\lambda_{12} + \mu_{12}) \frac{\partial}{\partial \theta_3} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta^2} + \frac{\partial u_3}{\partial \theta_3} \right) + \mu_{12} \left(\frac{\partial^2 u_3}{\partial \theta_1^2} + \frac{\partial^2 u_3}{\partial \theta_2^2} + \frac{\partial^2 u_3}{\partial \theta_3^2} \right) \\
 & + \frac{\xi_1}{2} \{ \bar{\lambda}_{12} \frac{\partial^2}{\partial \theta_3^2} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) + \bar{\mu}_{12} \frac{\partial}{\partial \theta_3} \left(\frac{\partial^2 u_3}{\partial \theta_1^2} + \frac{\partial^2 u_3}{\partial \theta_2^2} \right) \} \\
 & - \frac{\xi_1}{2} \bar{\mu}_{12} \left(\frac{\partial^2}{\partial \theta_3^2} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta^2} \right) + \frac{\partial}{\partial \theta_3} \left(\frac{\partial^2 u_3}{\partial \theta_1^2} + \frac{\partial^2 u_3}{\partial \theta_2^2} \right) \right) \\
 & - \frac{\xi_1^2}{3} \bar{\mu}_{12} \frac{\partial^2}{\partial \theta_3^2} \left(\frac{\partial^2 u_3}{\partial \theta_1^3} + \frac{\partial^2 u_3}{\partial \theta_2^3} \right) = 0
 \end{aligned}$$

or

$$\begin{aligned}
 & (\lambda_{12} + \mu_{12}) \frac{\partial}{\partial \theta_3} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} + \frac{\partial u_3}{\partial \theta_3} \right) + \mu_{12} \left(\frac{\partial^2 u_3}{\partial \theta_1^2} + \frac{\partial^2 u_3}{\partial \theta_2^2} + \frac{\partial^2 u_3}{\partial \theta_3^2} \right) \\
 & + \frac{\xi_1}{2} (\bar{\lambda}_{12} - \bar{\mu}_{12}) \frac{\partial^2}{\partial \theta_3^2} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) \\
 & - \frac{\xi_1^2}{3} \bar{\mu}_{12} \frac{\partial^2}{\partial \theta_3^2} \left(\frac{\partial^2 u_3}{\partial \theta_1^2} + \frac{\partial^2 u_3}{\partial \theta_2^2} \right) = 0
 \end{aligned} \tag{4.52}$$

For future reference we will also calculate σ_j for static problem in the absence of body forces. The preceding results are used in conjunction with equation (4.31) which has been simplified for such a case and reproduced before the equation (4.39). From (4.45) and (4.46) and (4.47) we have

$$\{mc_{\alpha j k l}^{(1)} + \frac{(1-m^2)}{m} c_{\alpha j k l}^{(2)}\} u_{k, \alpha} = \bar{\lambda}_{12} \delta_{\alpha j} u_{k, \alpha} + \bar{\mu}_{12} (u_{\alpha, j \alpha} + u_{j, \alpha \alpha}) \tag{4.53}$$

$$\{mc_{\alpha j k \beta}^{(1)} + \frac{1-m^3}{m^2} c_{\alpha j k \beta}^{(2)}\} u_{k, \beta \alpha} = \bar{\lambda}_{12} \delta_{\alpha j} u_{\beta, \beta \alpha} + \bar{\mu}_{12} (\delta_{j \beta} u_{\alpha, \beta \alpha} + u_{j, \alpha \alpha \beta}) \tag{4.54}$$

Using (4.36) we get

$$\begin{aligned}
 & \{mc_{j k l}^{(1)} + (1-m)c_{j k l}^{(2)}\} u_{k, l} = [m\lambda_1 + (1-m)\lambda_2] \delta_{3j} \delta_{kl} u_{k, l} \\
 & + [m\mu_1 + (1-m)\mu_2] [\delta_{3k} \delta_{jl} + \delta_{3l} \delta_{jk}] u_{k, l} = \lambda_{12} \delta_{3j} u_{k, k} + \mu_{12} (u_{3, j} + u_{j, 3})
 \end{aligned} \tag{4.55}$$

$$\begin{aligned}
 & \{mc_{j k \alpha}^{(1)} + \frac{(1-m^2)}{m} c_{j k \alpha}^{(2)}\} u_{k, \alpha} = \bar{\lambda}_{12} \delta_{3j} \delta_{k \alpha} u_{k, \alpha} + \bar{\mu}_{12} (\delta_{3k} \delta_{j \alpha} + \delta_{3 \alpha} \delta_{jk}) u_{k, \alpha} \\
 & = \bar{\lambda}_{12} \delta_{3j} u_{\alpha, \alpha} + \bar{\mu}_{12} \delta_{j \alpha} u_{3, \alpha}
 \end{aligned} \tag{4.56}$$

Substituting (4.53)-(4.56) in relation for σ_j we obtain

$$\begin{aligned}
 \sigma_j = & \lambda_{12} \delta_{3j} u_{k,k} + \mu_{12} (u_{3,j} + u_{j,3}) + \frac{\xi_1}{2} [\bar{\lambda}_{12} \delta_{3j} u_{\alpha,\alpha 3} + \bar{\mu}_{12} \delta_{j\alpha} u_{3,\alpha 3}] \\
 & - \frac{\xi_1}{2} [\bar{\lambda}_{12} \delta_{\alpha j} u_{k,k\alpha} + \bar{\mu}_{12} (u_{\alpha,j\alpha} + u_{j,\alpha\alpha})] \\
 & - \frac{1}{3} \xi_1^2 [\bar{\lambda}_{12} \delta_{\alpha j} u_{\beta,\beta\alpha 3} + \bar{\mu}_{12} (\delta_{j\beta} u_{\alpha,\beta\alpha 3} + u_{j,\alpha\alpha 3})] \quad (4.57)
 \end{aligned}$$

Writing down the components of σ_j separately we get

$$\begin{aligned}
 \sigma_1 = & \mu_{12} \left(\frac{\partial u_3}{\partial \theta_1} + \frac{\partial u_1}{\partial \theta_3} \right) + \frac{\xi_1}{2} \bar{\mu}_{12} \frac{\partial^2 u_3}{\partial \theta_1 \partial \theta_3} - \frac{\xi_1}{2} \bar{\lambda}_{12} \frac{\partial}{\partial \theta_1} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} + \frac{\partial u_3}{\partial \theta_3} \right) \\
 & - \frac{\xi_1}{2} \bar{\mu}_{12} \frac{\partial}{\partial \theta_1} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) - \frac{\xi_1}{2} \bar{\mu}_{12} \left(\frac{\partial^2 u_1}{\partial \theta_1^2} + \frac{\partial^2 u_1}{\partial \theta_2^2} \right) \\
 & - \frac{\xi_1^2}{3} \bar{\lambda}_{12} \frac{\partial^2}{\partial \theta_1 \partial \theta_3} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) \\
 & - \frac{\xi_1^2}{3} \bar{\mu}_{12} \frac{\partial^2}{\partial \theta_1 \partial \theta_3} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) - \frac{\xi_1^2}{3} \bar{\mu}_{12} \frac{\partial}{\partial \theta_3} \left(\frac{\partial^2 u_1}{\partial \theta_1^2} + \frac{\partial^2 u_1}{\partial \theta_2^2} \right) \\
 = & \mu_{12} \left(\frac{\partial u_1}{\partial \theta_3} + \frac{\partial u_3}{\partial \theta_1} \right) + \frac{\xi_1}{2} (\bar{\mu}_{12} - \bar{\lambda}_{12}) \frac{\partial^2 u_3}{\partial \theta_1 \partial \theta_3} - \frac{\xi_1}{2} (\bar{\lambda}_{12} + \bar{\mu}_{12}) \frac{\partial}{\partial \theta_1} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) \\
 & - \frac{\xi_1}{2} \bar{\mu}_{12} \left(\frac{\partial^2 u_1}{\partial \theta_1^2} + \frac{\partial^2 u_1}{\partial \theta_2^2} \right) - \frac{\xi_1^2}{3} (\bar{\lambda}_{12} + \bar{\mu}_{12}) \frac{\partial^2}{\partial \theta_1 \partial \theta_3} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) \\
 & - \frac{\xi_1^2}{3} \frac{\partial}{\partial \theta_3} \left(\frac{\partial^2 u_1}{\partial \theta_1^2} + \frac{\partial^2 u_1}{\partial \theta_2^2} \right) \bar{\mu}_{12} \quad (4.58)
 \end{aligned}$$

$$\sigma_2 = \mu_{12} \left(\frac{\partial u_2}{\partial \theta_3} + \frac{\partial u_3}{\partial \theta_2} \right) + \frac{\xi_1}{2} (\bar{\mu}_{12} - \bar{\lambda}_{12}) \frac{\partial^2 u_3}{\partial \theta_2 \partial \theta_3} - \frac{\xi_1}{2} (\bar{\lambda}_{12} + \bar{\mu}_{12}) \frac{\partial}{\partial \theta_2} \left(\frac{\partial u_1}{\partial \theta_2} + \frac{\partial u_2}{\partial \theta_2} \right)$$

$$\begin{aligned}
 & -\frac{\xi_1}{2} \bar{\mu}_{12} \left(\frac{\partial^2 u_2}{\partial \theta_1^2} + \frac{\partial^2 u_2}{\partial \theta_2^2} \right) - \frac{\xi_1^2}{3} (\bar{\lambda}_{12} + \bar{\mu}_{12}) \frac{\partial^2}{\partial \theta_2 \partial \theta_3} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) \\
 & - \frac{\xi_1^2}{3} \frac{\partial}{\partial \theta_3} \left(\frac{\partial^2 u_2}{\partial \theta_1^2} + \frac{\partial^2 u_2}{\partial \theta_2^2} \right) \bar{\mu}_{12}
 \end{aligned} \tag{4.59}$$

$$\begin{aligned}
 \sigma_3 &= \lambda_{12} u_{k,k} + 2\mu_{12} u_{3,3} + \frac{\xi_1}{2} (\bar{\lambda}_{12} u_{\alpha,\alpha 3}) - \frac{\xi_1}{2} \bar{\mu}_{12} (u_{\alpha,3\alpha} + u_{3,\alpha\alpha}) - \frac{\xi_1^3}{3} \bar{\mu}_{12} u_{3,\alpha\alpha 3} \\
 &= \lambda_{12} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} + \frac{\partial u_3}{\partial \theta_3} \right) + 2\mu_{12} \frac{\partial u_3}{\partial \theta_3} + \frac{\xi_1}{2} \bar{\lambda}_{12} \frac{\partial}{\partial \theta_3} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) \\
 &\quad - \frac{\xi_1}{2} \bar{\mu}_{12} \frac{\partial}{\partial \theta_3} \left(\frac{\partial u_1}{\partial \theta_2} + \frac{\partial u_2}{\partial \theta_2} \right) - \frac{\xi_1}{2} \bar{\mu}_{12} \left(\frac{\partial^2 u_3}{\partial \theta_1^2} + \frac{\partial^2 u_3}{\partial \theta_2^2} \right) \\
 &\quad - \frac{\xi_1^2}{3} \bar{\mu}_{12} \left(\frac{\partial^2 u_3}{\partial \theta_1^2} + \frac{\partial^2 u_3}{\partial \theta_2^2} \right)_{,3} \\
 \sigma_3 &= \lambda_{12} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) + (\lambda_{12} + 2\mu_{12}) \frac{\partial u_3}{\partial \theta_3} + \frac{\xi_1}{2} (\bar{\lambda}_{12} - \bar{\mu}_{12}) \frac{\partial}{\partial \theta_3} \left(\frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_2}{\partial \theta_2} \right) \\
 &\quad - \frac{\xi_1}{2} \bar{\mu}_{12} \left(\frac{\partial^2 u_3}{\partial \theta_1^2} + \frac{\partial^2 u_3}{\partial \theta_2^2} \right) - \frac{\xi_1^2}{3} \bar{\mu}_{12} \frac{\partial}{\partial \theta_3} \left(\frac{\partial^2 u_3}{\partial \theta_1^2} + \frac{\partial^2 u_3}{\partial \theta_2^2} \right)
 \end{aligned} \tag{4.60}$$

The constitutive relations (4.28) and (4.29) for τ_{ij} and $S_{\alpha j}$ for a flat composite are also simplified here for a composite laminate whose micro-structure is composed of two isotropic layers. Using (4.36) first we simplify different terms of the expressions (4.28) and (4.29)

$$\begin{aligned}
 \{m c_{ijk}^{(1)} + (1-m) c_{ijk}^{(2)}\} u_{k,l} &= \{m \lambda_1 + (1-m) \lambda_2\} \delta_{ij} \delta_{kl} u_{k,l} \\
 &\quad + \{m \mu_1 + (1-m) \mu_2\} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} + u_{k,l}) \\
 &= \lambda_{12} \delta_{ij} u_{k,k} + \mu_{12} (u_{i,j} + u_{j,i})
 \end{aligned} \tag{4.61}$$

where (30.47)_{1,2} are used.

$$\begin{aligned} \{mc_{\alpha j k l}^{(1)} + \frac{(1-m^2)}{m} c_{\alpha j k l}^{(2)}\} u_{k,l} &= \bar{\lambda}_{12} \delta_{\alpha j} \delta_{k l} u_{k,l} + \bar{\mu}_{12} (\delta_{\alpha k} \delta_{j l} + \delta_{\alpha l} \delta_{j k}) u_{k,l} \\ &= \bar{\lambda}_{12} \delta_{\alpha j} u_{k,k} + \bar{\mu}_{12} (u_{\alpha,j} + u_{j,\alpha}) \end{aligned} \quad (4.62)$$

$$\begin{aligned} \{mc_{ij \alpha 3}^{(1)} + \frac{(1-m^2)}{m} c_{ij \alpha 3}^{(2)}\} u_{k,\alpha 3} &= \bar{\lambda}_{12} \delta_{ij} \delta_{\alpha 3} u_{k,\alpha 3} + \bar{\mu}_{12} (\delta_{ik} \delta_{j \alpha} + \delta_{i \alpha} \delta_{j k}) u_{k,\alpha 3} \\ &= \bar{\lambda}_{12} \delta_{ij} u_{\alpha,\alpha 3} + \bar{\mu}_{12} (\delta_{j \alpha} u_{i,\alpha 3} + \delta_{i \alpha} u_{j,\alpha 3}) \end{aligned} \quad (4.63)$$

$$\begin{aligned} \{mc_{\alpha j k \beta}^{(1)} + \frac{1-m^3}{m^2} c_{\alpha j k \beta}^{(2)}\} u_{k,\beta 3} &= \bar{\lambda}_{12} \delta_{\alpha j} \delta_{k \beta} u_{k,\beta 3} + \bar{\mu}_{12} (\delta_{\alpha k} \delta_{j \beta} + \delta_{\alpha \beta} \delta_{j k}) u_{k,\beta 3} \\ &= \bar{\lambda}_{12} \delta_{\alpha j} u_{\beta,\beta 3} + \bar{\mu}_{12} (\delta_{j \beta} u_{\alpha,\beta 3} + u_{j,\alpha 3}) \end{aligned} \quad (4.64)$$

Substituting (4.61)-(4.64) in (4.28) and (4.29) we obtain

$$\begin{aligned} \tau_{ij} &= \lambda_{12} \delta_{ij} u_{k,k} + \mu_{12} (u_{i,j} + u_{j,i}) \\ &+ \frac{\xi_1}{2} [\bar{\lambda}_{12} \delta_{ij} u_{\alpha,\alpha 3} + \bar{\mu}_{12} (\delta_{j \alpha} u_{i,\alpha 3} + \delta_{i \alpha} u_{j,\alpha 3})] \end{aligned} \quad (4.65)$$

$$\begin{aligned} S_{\alpha j} &= \frac{\xi_1}{2} [\bar{\lambda}_{12} \delta_{\alpha j} u_{k,k} + \bar{\mu}_{12} (u_{\alpha,j} + u_{j,\alpha})] \\ &+ \frac{1}{3} \xi_1^2 [\bar{\lambda}_{12} \delta_{\alpha j} u_{\beta,\beta 3} + \bar{\mu}_{12} (\delta_{j \beta} u_{\alpha,\beta 3} + u_{j,\alpha 3})] \end{aligned} \quad (4.66)$$

5.0 LINEAR CONSTITUTIVE RELATIONS FOR A MULTI-CONSTITUENT COMPOSITE

In this section we assume that the representative micro-structure is composed of n layers with different constituents. For such a micro-structure we let

$$\tau_{(\alpha)}^{*ij} = c_{(\alpha)}^{ijkL} \gamma_{kl}^* \quad (\alpha = 1, 2, \dots, n) \quad (5.1)$$

where $c_{(\alpha)}^{ijkL}$ ($\alpha = 1, \dots, n$) are material constants in the associated layers. As before the variable ξ is designated to change across the micro-structure whose thickness is assumed to be ξ_n . It should be noted that although the micro-structure is composed of n layers, ξ_n is still supposed to be a very small number. The range of variation of ξ in the l^{th} layer of the microstructure is from ξ_{l-1} to ξ_l where $l = 1, \dots, n$ and $\xi_0 = 0$. This convention is adopted due to its agreement with the special case of a two-layered micro-structure which was studied before. We further define $(n-1)$ constants m_1, \dots, m_{n-1} according to the following relations

$$\xi_l = m_l \xi_n \quad (l = 1, \dots, n-1) \quad (5.2)$$

As a result of this definition the thickness of the l^{th} layer of the micro-structure is equal to $(m_l - m_{l-1})\xi_n$ where $l = 1, \dots, n$ and $m_0 = 0$, $m_n = 1$. The *composite stress vector* T^i and the *composite couple stress* S^α and other quantities are obviously defined over the whole thickness of the micro-structure. For example we have

$$T^i = \frac{1}{\xi_n} \int_0^{\xi_n} T^{*i} d\xi \quad (5.3)$$

$$S^\alpha = \frac{1}{\xi_n} \int_0^{\xi_n} \xi T^{*\alpha} d\xi \quad (5.4)$$

In order to derive appropriate constitutive relations we make the following definition.

Let

$$a = a_l \quad \text{for} \quad \xi_{l-1} < \xi < \xi_l \quad (l = 1, 2, \dots, n) \quad (5.5)$$

where a_l 's are constants and $\xi_0 = 0$ as noted earlier. The function a is piecewise continuous for $\xi \in (0, \xi_n)$ and we can evaluate the following integral

$$\frac{1}{\xi_n} \int_0^{\xi_n} \xi^k a \, d\xi = \frac{1}{\xi_n} \sum_{l=1}^n \int_{\xi_{l-1}}^{\xi_l} \xi^k a_l \, d\xi = \frac{1}{\xi_n} \sum_{l=1}^n \frac{a_l}{k+1} (\xi_l^{k+1} - \xi_{l-1}^{k+1}) \quad (k \neq -1) \quad (5.6)$$

Using definitions (5.2) we further simplify (5.6)

$$\frac{1}{\xi_n} \int_0^{\xi_n} \xi^k a \, d\xi = \frac{1}{\xi_n} \frac{1}{k+1} \sum_{l=1}^n a_l [m_l^{k+1} - m_{l-1}^{k+1}] \xi_n^{k+1} = \frac{\xi_n^k}{k+1} \sum_{l=1}^n a_l (m_l^{k+1} - m_{l-1}^{k+1}) \quad (k \neq -1) \quad (5.7)$$

where $m_0 = 0$ and $m_n = 1$. To simplify the final results in constitutive relations we first notice that the integrals which appear in these equations are the weighted averages of the constitutive coefficients. So we adopt the following definition

$$I^{(k)pqrs} \triangleq \frac{1}{\xi_n} \int_0^{\xi_n} \xi^k c^{pqrs} \, d\xi \quad (5.8)$$

which by (5.7) is seen to be equal to

$$I^{(k)pqrs} = \frac{1}{k+1} \xi_n^k \sum_{l=1}^n c_{(l)}^{pqrs} (m_l^{k+1} - m_{l-1}^{k+1}) \quad (5.9)$$

We use the same contravariant or covariant index notations for I and c . However, the weighting number k is always written as a superscript in parentheses. Whenever the covariant components of constitutive coefficients are used, the layer index (l) is also written as a superscript in parentheses. Recall (3.22) which for the present situation can be written as

$$T^i = \gamma_{kl} \frac{1}{\xi_n} \int_0^{\xi_n} c^{ijkl} g^{*1/2} g_j^* \, d\xi + \kappa_{la} \frac{1}{\xi_n} \int_0^{\xi_n} \xi c^{ijla} g^{*1/2} g_j^* \, d\xi \quad (5.10)$$

Combining (3.4) and (2.9)₃ we obtain

$$g^{*1/2} g_{\beta}^* = g^{1/2} (1 + \frac{\xi \Delta}{2g}) (g_{\beta} + \xi \lambda_{\beta}^j g_j) \quad (5.11)$$

and (3.5) reads

$$g^{*1/2} g_3^* = g^{1/2} (1 + \frac{\xi \Delta}{2g}) g_3 \quad (5.12)$$

The first integral in relation (5.10) is simplified using (5.11) and (5.12). We have

$$\begin{aligned} \frac{1}{\xi_n} \int_0^{\xi_n} c^{ijkl} g^{*1/2} g_j^* d\xi &= \frac{1}{\xi_n} \int_0^{\xi_n} (c^{i\beta k l} g^{*1/2} g_{\beta}^* + c^{i3 k l} g^{*1/2} g_3^*) d\xi \\ &= g^{1/2} g_{\beta} \frac{1}{\xi_n} \int_0^{\xi_n} (1 + \frac{\xi \Delta}{2g}) c^{i\beta k l} d\xi + g^{1/2} \lambda_{\beta}^j g_j \frac{1}{\xi_n} \int_0^{\xi_n} \xi (1 + \frac{\xi \Delta}{2g}) c^{i\beta k l} d\xi \\ &\quad + g^{1/2} g_3 \frac{1}{\xi_n} \int_0^{\xi_n} (1 + \frac{\xi \Delta}{2g}) c^{i3 k l} d\xi \\ &= g^{1/2} g_j \left(\frac{1}{\xi_n} \int_0^{\xi_n} (1 + \frac{\xi \Delta}{2g}) c^{ijk l} d\xi + \lambda_{\beta}^j \frac{1}{\xi_n} \int_0^{\xi_n} \xi (1 + \frac{\xi \Delta}{2g}) c^{i\beta k l} d\xi \right) \quad (5.13) \end{aligned}$$

Now we use definition (5.8) to simplify (5.13). The following expression would be the result

$$1^{st} \text{ term in (5.10)} = g^{1/2} g_j \{ I^{(0)ijk l} + \frac{\Delta}{2g} I^{(1)ijk l} + \lambda_{\beta}^j (I^{(1)i\beta k l} + \frac{\Delta}{2g} I^{(2)i\beta k l}) \} \quad (5.14)$$

The second integral in (5.10) is also simplified similarly

$$\begin{aligned} \frac{1}{\xi_n} \int_0^{\xi_n} \xi c^{ij\alpha} g^{*1/2} g_j^* d\xi &= \frac{1}{\xi_n} \int_0^{\xi_n} \xi g^{*1/2} (c^{i\beta \alpha} g_{\beta}^* + c^{i3 \alpha} g_3^*) d\xi \\ &= \frac{1}{\xi_n} \int_0^{\xi_n} \xi g^{1/2} (1 + \frac{\xi \Delta}{2g}) c^{i\beta \alpha} g_{\beta} d\xi + \frac{1}{\xi_n} \int_0^{\xi_n} \xi^2 g^{1/2} (1 + \frac{\xi \Delta}{2g}) \lambda_{\beta}^j c^{i\beta \alpha} g_j d\xi \\ &\quad + \frac{1}{\xi_n} \int_0^{\xi_n} \xi g^{1/2} (1 + \frac{\xi \Delta}{2g}) c^{i3 \alpha} g_3 d\xi \end{aligned}$$

$$= g^{1/2} g_j \left\{ \frac{1}{\xi_n} \int_0^{\xi_n} \xi \left(1 + \frac{\xi \Delta}{2g} \right) c^{ij\alpha} d\xi + \lambda_\beta^j \frac{1}{\xi_n} \int_0^{\xi_n} \xi^2 \left(1 + \frac{\xi \Delta}{2g} \right) c^{i\beta\alpha} d\xi \right\} \quad (5.15)$$

which again by using (5.8) is simplified to

$$2^{\text{nd}} \text{ term in (5.10)} = g^{1/2} g_j \left\{ I^{(1)ij\alpha} + \frac{\Delta}{2g} I^{(1)ij\alpha} + \lambda_\beta^j I^{(2)i\beta\alpha} + \lambda_\beta^j \frac{\Delta}{2g} I^{(3)i\beta\alpha} \right\} \quad (5.16)$$

Substituting (5.14) and (5.16) in (5.10) we get the following constitutive relation for T^i

$$T^i = \left\{ [I^{(0)ijk} + \frac{\Delta}{2g} I^{(1)ijk} + \lambda_\beta^j (I^{(1)i\beta k} + \frac{\Delta}{2g} I^{(2)i\beta k})] \gamma_{kl} \right. \\ \left. + [I^{(1)ij\alpha} + \frac{\Delta}{2g} I^{(2)ij\alpha} + \lambda_\beta^j (I^{(2)i\beta\alpha} + \frac{\Delta}{2g} I^{(3)i\beta\alpha})] \kappa_{l\alpha} \right\} g^{1/2} g_j \quad (5.17)$$

The expression inside the bracket is obviously the constitutive relation for τ^{ij} . Hence

$$\tau^{ij} = [I^{(0)ijk} + \frac{\Delta}{2g} I^{(1)ijk} + \lambda_\beta^j (I^{(1)i\beta k} + \frac{\Delta}{2g} I^{(2)i\beta k})] \gamma_{kl} \\ + [I^{(1)ij\alpha} + \frac{\Delta}{2g} I^{(2)ij\alpha} + \lambda_\beta^j (I^{(2)i\beta\alpha} + \frac{\Delta}{2g} I^{(3)i\beta\alpha})] \kappa_{l\alpha} \quad (5.18)$$

The same steps are followed to derive the constitutive relation for the *composite couple stress* S^α . By (3.27) we have

$$S^\alpha = \gamma_{kl} \frac{1}{\xi_n} \int_0^{\xi_n} \xi g^{*1/2} c^{\alpha jkl} g_j^* d\xi + \kappa_\beta \frac{1}{\xi_n} \int_0^{\xi_n} \xi^2 c^{\alpha j\beta} g^{*1/2} g_j^* d\xi \quad (5.19)$$

which can be reduced to the following form by exactly using the same procedure

$$S^\alpha = \left\{ I^{(1)\alpha jkl} + \frac{\Delta}{2g} I^{(2)\alpha jkl} + \lambda_\gamma^j (I^{(2)\alpha\gamma k} + \frac{\Delta}{2g} I^{(3)\alpha\gamma k}) \right\} \gamma_{kl} \\ + \left\{ I^{(2)\alpha j\beta} + \frac{\Delta}{2g} I^{(3)\alpha j\beta} + \lambda_\gamma^j (I^{(3)\alpha\gamma\beta} + \frac{\Delta}{2g} I^{(4)\alpha\gamma\beta}) \right\} \kappa_\beta \} g^{1/2} g_j \quad (5.20)$$

Subsequently the constitutive relation for $S^{\alpha j}$ would be

$$\begin{aligned} S^{\alpha j} = & [I^{(1)\alpha jkl} + \frac{\Delta}{2g} I^{(2)\alpha jkl} + \lambda_{\gamma}^j (I^{(2)\alpha \gamma kl} + \frac{\Delta}{2g} I^{(3)\alpha \gamma kl})] \gamma_{kl} \\ & + [I^{(2)\alpha j l \beta} + \frac{\Delta}{2g} I^{(3)\alpha j l \beta} + \lambda_{\gamma}^j (I^{(3)\alpha \gamma l \beta} + \frac{\Delta}{2g} I^{(4)\alpha \gamma l \beta})] \kappa_{l \beta} \end{aligned} \quad (5.21)$$

For small deformations of a composite with initially flat plies, the foregoing equations are further simplified. The resulting relations are recorded here:

$$\tau^{ij} = I^{(0)ijkL} \gamma_{kl} + I^{(1)ijl\alpha} \kappa_{l\alpha} \quad (5.22)$$

$$S^{\alpha j} = I^{(1)\alpha jkl} \gamma_{kl} + I^{(2)\alpha j l \beta} \kappa_{l \beta} \quad (5.23)$$

with no distinction between contravariant and covariant components. In terms of displacement vector u and its gradients these equations can be written as follows:

$$\tau_{ij} = I_{ijkL}^{(0)} u_{k,L} + I_{ijl\alpha}^{(1)} u_{l,\alpha} \quad (5.24)$$

$$S_{\alpha j} = I_{\alpha jkl}^{(1)} u_{k,L} + I_{\alpha j l \beta}^{(2)} u_{l,\beta} \quad (5.25)$$

Using (5.9) the constitutive coefficients are written in the expanded form

$$I_{ijkL}^{(0)} = \sum_{r=1}^n c_{ijkL}^{(r)} (m_r - m_{r-1})$$

$$I_{ijkL}^{(1)} = \frac{1}{2} \xi_n \sum_{r=1}^n c_{ijkL}^{(r)} (m_r^2 - m_{r-1}^2) \quad (5.26)$$

$$I_{ijkL}^{(2)} = \frac{1}{2} \xi_n^2 \sum_{r=1}^n c_{ijkL}^{(r)} (m_r^3 - m_{r-1}^3)$$

To recapitulate, $d_{ijkL}^{(r)}$ ($r = 1, \dots, n$) are the constitutive coefficients of the micro-structure layers and m_r 's ($r = 1, \dots, n-1$) are dimensionless constants related to the thicknesses of different layers with $m_0 = 0$ and $m_n = 1$.

If the micro-structure is composed of n isotropic layers, we can write (5.26) in terms of the Lamé's constants of various layers. In fact, we have

$$c_{ijkl}^{(r)} = \lambda_{(r)} \delta_{ij} \delta_{kl} + \mu_{(r)} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad r = 1, \dots, n \quad (5.27)$$

For such a case, the relations (5.24) and (5.25) are written in expanded forms as follows

$$\begin{aligned} \tau_{ij} = & u_{k,k} \delta_{ij} \sum_{r=1}^n \lambda_{(r)} (m_r - m_{r-1}) + (u_{i,j} + u_{j,i}) \sum_{r=1}^n \mu_{(r)} (m_1 - m_{r-1}) \\ & + \frac{\xi_n}{2} [\delta_{ij} \delta_{l\alpha} u_{l,\alpha 3} \sum_{r=1}^n \lambda_{(r)} (m_r^2 - m_{r-1}^2) + (\delta_{il} \delta_{j\alpha} + \delta_{i\alpha} \delta_{jl}) \\ & \times u_{l,\alpha 3} \sum_{r=1}^n \mu_{(r)} (m_r^2 - m_{r-1}^2)] \end{aligned}$$

or

$$\begin{aligned} \tau_{ij} = & u_{k,k} \delta_{ij} \sum_{r=1}^n \lambda_{(r)} \Delta m_r + (u_{i,j} + u_{j,i}) \sum_{r=1}^n \mu_{(r)} \Delta m_r + \frac{\xi_n}{2} \delta_{ij} u_{\alpha\alpha 3} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^2 \\ & + \frac{\xi_n}{2} (u_{i,\alpha 3} \delta_{j\alpha} + u_{j,\alpha 3} \delta_{i\alpha}) \sum_{r=1}^n \mu_{(r)} \Delta m_r^2 \end{aligned} \quad (5.28)$$

$$\begin{aligned} S_{\alpha j} = & \frac{1}{2} \xi_n [\delta_{\alpha j} \delta_{kl} u_{k,l} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^2 + (\delta_{\alpha k} \delta_{jl} + \delta_{\alpha l} \delta_{jk}) u_{k,l} \sum_{r=1}^n \mu_{(r)} \Delta m_r^2] \\ & + \frac{1}{3} \xi_n^2 [\delta_{\alpha j} \delta_{\beta\beta} u_{l,\beta 3} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^3 + (\delta_{\alpha l} \delta_{j\beta} + \delta_{\alpha\beta} \delta_{jl}) u_{l,\beta 3} \sum_{r=1}^n \mu_{(r)} \Delta m_r^3] \end{aligned}$$

or

$$\begin{aligned} S_{\alpha j} = & \frac{1}{2} \xi_n [\delta_{\alpha j} u_{k,k} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^2 + (u_{\alpha,j} + u_{j,\alpha}) \sum_{r=1}^n \mu_{(r)} \Delta m_r^2] \\ & + \frac{1}{3} \xi_n^2 [\delta_{\alpha j} u_{\beta,\beta 3} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^3 + (\delta_{j\beta} u_{\alpha,\beta 3} + u_{j,\alpha 3}) \sum_{r=1}^n \mu_{(r)} \Delta m_r^3] \end{aligned} \quad (5.29)$$

where for brevity we have introduced

$$\Delta m_r^p = m_r^p - m_{r-1}^p \quad r = 1, \dots, n \quad (5.30)$$

and in the above relations $p = 1, 2, 3$.

In order to obtain the complete field equations for a linear elastic composite whose micro-structures comprise n layers we should substitute the constitutive relations (5.24) and (5.25) in the equations of motion (2.123) and (2.124). However before that we should obtain appropriate expressions for ρ_o , $\rho_o z^1$ and $\rho_o z^2$. We assume that the micro-structure layers are homogeneous with densities $\rho_o^{(r)}$ ($r = 1, \dots, n$) in the reference configuration. Recalling (2.45) and (2.46) we can write

$$\rho g^{1/2} = \rho_o G^{1/2} = \frac{1}{\xi_n} \int_0^{\xi_n} \rho_o^* G^{*1/2} d\xi \quad (5.31)$$

where

$$\rho_o^* = \rho_o^{(r)} \quad \text{for} \quad \xi_{r-1} < \xi < \xi_r \quad (r = 1, \dots, n) \quad (5.32)$$

and

$$\xi_o = 0$$

Using (4.2), (5.32) and (5.7) in (5.31) we conclude

$$\rho_o = \frac{1}{\xi_n} \int_0^{\xi_n} \left(1 + \frac{\xi \Delta}{2G}\right) \rho_o^* d\xi = \sum_{r=1}^n \rho_o^{(r)} \Delta m_r + \frac{\Delta \xi_n}{4G} \sum_{r=1}^n \rho_o^{(r)} \Delta m_r^2 \quad (5.33)$$

We will proceed similarly to calculate $\rho_o z^1$ and $\rho_o z^2$ using their definitions for the present situation. By (2.42) we have

$$\rho g^{1/2} z^1 = \rho_o G^{1/2} z^1 = \frac{1}{\xi_n} \int_0^{\xi_n} \xi \rho_o^* G^{*1/2} d\xi \quad (5.34)$$

$$\rho g^{1/2} z^2 = \rho_0 G^{1/2} z^2 = \frac{1}{\xi_n} \int_0^{\xi_n} \xi^2 \rho_0^* G^{1/2} d\xi \quad (5.35)$$

Similar to what was done in the derivation of (5.33) we write

$$\rho_0 z^1 = \frac{1}{\xi_n} \int_0^{\xi_n} \xi \left(1 + \frac{\xi \Delta}{2G}\right) \rho_0^* d\xi = \frac{1}{2} \xi_n \sum_{r=1}^n \rho_0^{(r)} \Delta m_r^2 + \frac{\Delta \xi_n^2}{6G} \sum_{r=1}^n \rho_0^{(r)} \Delta m_r^3 \quad (5.36)$$

and

$$\rho_0 z^2 = \frac{1}{\xi_n} \int_0^{\xi_n} \xi^2 \left(1 + \frac{\xi \Delta}{2G}\right) \rho_0^* d\xi = \frac{1}{3} \xi_n^2 \sum_{r=1}^n \rho_0^{(r)} \Delta m_r^3 + \frac{\Delta \xi_n^3}{8G} \sum_{r=1}^n \rho_0^{(r)} \Delta m_r^4 \quad (5.37)$$

For a composite with initially flat plies, equations (5.35), (5.36) and (5.37) are reduced respectively to

$$\rho_0 = \sum_{r=1}^n \rho_0^{(r)} \Delta m_r \quad (5.38)$$

$$\rho_0 z^1 = \frac{1}{2} \xi_n \sum_{r=1}^n \rho_0^{(r)} \Delta m_r^2 \quad (5.39)$$

$$\rho_0 z^2 = \frac{1}{3} \xi_n^2 \sum_{r=1}^n \rho_0^{(r)} \Delta m_r^3 \quad (5.40)$$

Now we can substitute (5.24), (5.25), (5.38)-(5.40) and (4.23) in (2.123) and (2.124) to derive the linear equations of motion for a flat composite. The resulting equations are recorded below:

$$\begin{aligned} I_{\alpha j k l}^{(0)} u_{k, l \alpha} + I_{\alpha j \beta}^{(1)} u_{l, \alpha \beta} + b_j \sum_{r=1}^n \rho_0^{(r)} \Delta m_r + \sigma_{j, 3} \\ = \ddot{u}_j \sum_{r=1}^n \rho_0^{(r)} \Delta m_r + \frac{1}{2} \xi_n \ddot{u}_{j, 3} \sum_{r=1}^n \rho_0^{(r)} \Delta m_r^2 \end{aligned} \quad (5.41)$$

$$I_{\alpha j k l}^{(1)} u_{k, \alpha} + I_{\alpha j \beta}^{(2)} u_{l, \alpha \beta} + \sigma_j - I_{3 j k l}^{(0)} u_{k, l} - I_{3 j \beta}^{(1)} u_{l, \beta} + c_j \sum_{r=1}^n \rho_o^{(r)} \Delta m_r = \frac{1}{2} \xi_n \ddot{u}_j \sum_{r=1}^n \rho_o^{(r)} \Delta m_r^2 + \frac{1}{3} \xi_n^2 \ddot{u}_{j,3} \sum_{r=1}^n \rho_o^{(r)} \Delta m_r^3 \quad (5.42)$$

The appropriate differential equation for the displacement vector u is obtained by eliminating σ_j from equations (5.41) and (5.42). For static problems with no body force these equations reduce to

$$I_{\alpha j k l}^{(0)} u_{k, \alpha} + I_{\alpha j k \beta}^{(1)} u_{k, \alpha \beta} + \sigma_{j,3} = 0 \quad (5.43)$$

$$I_{\alpha j k l}^{(1)} u_{k, \alpha} + I_{\alpha j k \beta}^{(2)} u_{k, \alpha \beta} + \sigma_j - I_{3 j k l}^{(0)} u_{k, l} - I_{3 j k \beta}^{(1)} u_{k, \beta} = 0 \quad (5.44)$$

Eliminating σ_j between these equations, we get

$$I_{\alpha j k l}^{(0)} u_{k, \alpha} + I_{\alpha j k \beta}^{(1)} u_{k, \alpha \beta} + I_{3 j k l}^{(0)} u_{k, l} + I_{3 j k \beta}^{(1)} u_{k, \beta} - I_{\alpha j k l}^{(1)} u_{k, \alpha} - I_{\alpha j k \beta}^{(2)} u_{k, \alpha \beta} = 0$$

or

$$I_{i j k l}^{(0)} u_{k, i} + I_{i j k \beta}^{(1)} u_{k, i \beta} - I_{\alpha j k l}^{(1)} u_{k, \alpha} - I_{\alpha j k \beta}^{(2)} u_{k, \alpha \beta} = 0 \quad (5.45)$$

This is a fourth order partial differential equation for the displacement vector u . The constitutive coefficients $I_{i j k l}^{(r)}$ ($r = 0, 1, 2$) are already written in expanded forms in Eqs. (5.26).

For a composite laminate whose micro-structure is composed of n isotropic layers we use (5.27) and (5.26) to rewrite (5.45). The result is

$$\begin{aligned} & \delta_{ij} \delta_{kl} u_{k, i} \sum_{r=1}^n \lambda_{(r)} \Delta m_r + (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) u_{k, i} \sum_{r=1}^n \mu_{(r)} \Delta m_r \\ & + \frac{1}{2} \xi_n [\delta_{ij} \delta_{k\beta} u_{k, i\beta} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^2 + (\delta_{ik} \delta_{j\beta} + \delta_{i\beta} \delta_{jk}) u_{k, i\beta} \sum_{r=1}^n \mu_{(r)} \Delta m_r^2] \\ & - \frac{1}{2} \xi_n [\delta_{\alpha j} \delta_{kl} u_{k, \alpha} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^2 + (\delta_{\alpha k} \delta_{jl} + \delta_{\alpha l} \delta_{jk}) u_{k, \alpha} \sum_{r=1}^n \mu_{(r)} \Delta m_r^2] \end{aligned}$$

$$-\frac{1}{3} \xi_n^2 [\delta_{\alpha j} \delta_{k\beta} u_{k,\alpha\beta 33} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^3 + (\delta_{\alpha k} \delta_{j\beta} + \delta_{\alpha\beta} \delta_{jk}) u_{k,\alpha\beta 33} \sum_{r=1}^n \mu_{(r)} \Delta m_r^3] = 0$$

or

$$\begin{aligned} & u_{i,ji} \left(\sum_{r=1}^n \lambda_{(r)} \Delta m_r + \sum_{r=1}^n \mu_{(r)} \Delta m_r \right) + u_{j,ii} \sum_{r=1}^n \mu_{(r)} \Delta m_r \\ & + \frac{1}{2} \xi_n [u_{\beta j\beta 3} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^2 + (\delta_{j\beta} u_{i,i\beta 3} + u_{j,\beta\beta 3}) \sum_{r=1}^n \mu_{(r)} \Delta m_r^2] \\ & - \frac{1}{2} \xi_n [\delta_{\alpha j} u_{k,k\alpha 3} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^2 + (u_{\alpha,j\alpha 3} + u_{j,\alpha\alpha 3}) \sum_{r=1}^n \mu_{(r)} \Delta m_r^2] \\ & - \frac{1}{3} \xi_n^2 [\delta_{\alpha j} u_{\beta,\alpha\beta 33} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^3 + (\delta_{j\beta} u_{\alpha,\alpha\beta 33} + u_{j,\alpha\alpha 33}) \sum_{r=1}^n \mu_{(r)} \Delta m_r^3] \\ & = u_{i,ji} \sum_{r=1}^n (\lambda_{(r)} + \mu_{(r)}) \Delta m_r + u_{j,ii} \sum_{r=1}^n \mu_{(r)} \Delta m_r \\ & + \frac{1}{2} \xi_n (u_{\alpha,j\alpha 3} - \delta_{\alpha j} u_{i,i\alpha 3}) \sum_{r=1}^n [\lambda_{(r)} - \mu_{(r)}] \Delta m_r^2 \\ & - \frac{1}{3} \xi_n^2 \{ \delta_{\alpha j} u_{\beta,\alpha\beta 33} \sum_{r=1}^n [\lambda_{(r)} + \mu_{(r)}] \Delta m_r^3 + u_{j,\alpha\alpha 33} \sum_{r=1}^n \mu_{(r)} \Delta m_r^3 \} = 0 \quad (5.46) \end{aligned}$$

There are three partial differential equations of fourth order for displacement vector u and we can write them in two separate sets, one for $j = \gamma$ and the other for $j = 3$. For $j = \gamma$ we obtain

$$\begin{aligned} & u_{i,\gamma i} \sum_{r=1}^n (\lambda_{(r)} + \mu_{(r)}) \Delta m_r + u_{\gamma,ii} \sum_{r=1}^n \mu_{(r)} \Delta m_r \\ & - \frac{1}{2} \xi_n u_{3,\gamma 33} \sum_{r=1}^n (\lambda_{(r)} - \mu_{(r)}) \Delta m_r^2 \\ & - \frac{1}{3} \xi_n^2 \{ u_{\beta,\gamma\beta 33} \sum_{r=1}^n (\lambda_{(r)} + \mu_{(r)}) \Delta m_r^3 + u_{\gamma,\alpha\alpha 33} \sum_{r=1}^n \mu_{(r)} \Delta m_r^3 \} = 0 \quad (5.47) \end{aligned}$$

and for $j = 3$ we get

$$u_{i,3} \sum_{r=1}^n (\lambda_{(r)} + \mu_{(r)}) \Delta m_r + u_{3,ii} \sum_{r=1}^n \mu_{(r)} \Delta m_r + \frac{1}{2} \xi_n u_{\alpha, \alpha 33} \sum_{r=1}^n (\lambda_{(r)} - \mu_{(r)}) \Delta m_r^2 - \frac{1}{3} \xi_n^2 u_{3, \alpha \alpha 33} \sum_{r=1}^n \mu_{(r)} \Delta m_r^3 = 0 \quad (5.48)$$

For the special case where $n = 2$ these equations reduce to the equations (4.50)-(4.52) derived before for a bi-laminate composite. From (5.44) we can also calculate σ_j

$$\sigma_j = I_{3jk}^{(0)} u_{k,l} - I_{\alpha jkl}^{(1)} u_{k,l\alpha} + I_{3jk\beta}^{(1)} u_{k,\beta 3} - I_{\alpha jk\beta}^{(2)} u_{k,\alpha \beta 3} \quad (5.49)$$

which for the isotropic case reduces to

$$\begin{aligned} \sigma_j = & (\delta_{3j} \delta_{kl} u_{k,l}) \sum_{r=1}^n \lambda_{(r)} \Delta m_r + (\delta_{3k} \delta_{jl} + \delta_{3l} \delta_{jk}) u_{k,l} \sum_{r=1}^n \mu_{(r)} \Delta m_r \\ & - \frac{\xi_n}{2} [\delta_{\alpha j} \delta_{kl} u_{k,l\alpha} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^2 + (\delta_{\alpha k} \delta_{jl} + \delta_{\alpha l} \delta_{jk}) u_{k,l\alpha} \sum_{r=1}^n \mu_{(r)} \Delta m_r^2] \\ & + \frac{\xi_n}{2} [\delta_{3j} \delta_{k\beta} u_{k,\beta 3} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^2 + (\delta_{3\beta} \delta_{jk} + \delta_{3k} \delta_{j\beta}) u_{k,\beta 3} \sum_{r=1}^n \mu_{(r)} \Delta m_r^2] \\ & - \frac{1}{3} \xi_n^2 [\delta_{\alpha j} \delta_{k\beta} u_{k,\alpha \beta 3} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^3 + (\delta_{\alpha k} \delta_{j\beta} + \delta_{\alpha \beta} \delta_{jk}) u_{k,\alpha \beta 3} \sum_{r=1}^n \mu_{(r)} \Delta m_r^3] \end{aligned} \quad (5.50)$$

The equations (5.50) are written in two separate sets. For $j = \gamma$ we get

$$\begin{aligned} \sigma_\gamma = & (u_{\gamma,3} + u_{3,\gamma}) \sum_{r=1}^n \mu_{(r)} \Delta m_r + \frac{\xi_n}{2} u_{3,\gamma\beta} \sum_{r=1}^n [\mu_{(r)} - \lambda_{(r)}] \Delta m_r^2 \\ & - \frac{\xi_n}{2} [u_{\alpha,\alpha\gamma} \sum_{r=1}^n (\lambda_{(r)} + \mu_{(r)}) \Delta m_r^2 + u_{\gamma,\alpha\alpha} \sum_{r=1}^n \mu_{(r)} \Delta m_r^2] \\ & - \frac{1}{3} \xi_n^2 [u_{\alpha,\alpha\gamma\beta} \sum_{r=1}^n (\lambda_{(r)} + \mu_{(r)}) \Delta m_r^3 + u_{\gamma,\alpha\alpha\beta} \sum_{r=1}^n \mu_{(r)} \Delta m_r^3] \end{aligned} \quad (5.51)$$

and for $j = 3$ we obtain

$$\begin{aligned}
 \sigma_3 = & u_{k,k} \sum_{r=1}^n \lambda_{(r)} \Delta m_r + 2u_{3,3} \sum_{r=1}^n \mu_{(r)} \Delta m_r \\
 & - \frac{\xi_n}{2} (u_{\alpha,3\alpha} + u_{3,\alpha\alpha}) \sum_{r=1}^n \mu_{(r)} \Delta m_r^2 + \frac{\xi_n}{2} u_{\alpha,\alpha 3} \sum_{r=1}^n \lambda_{(r)} \Delta m_r^2 \\
 & - \frac{1}{3} \xi_n^2 u_{3,\alpha\alpha 3} \sum_{r=1}^n \mu_{(r)} \Delta m_r^3
 \end{aligned} \tag{5.52}$$

For the special case of a laminate with two layer micro-structure ($n=2$) these equations reduce to (4.58)-(4.60).

6.0 DEVELOPMENT OF THERMO-MECHANICAL THEORY FOR COMPOSITE LAMINATES

In order to develop a thermo-mechanical theory for the composite laminates, we begin by writing down the local balance of energy and the Clausius-Duhem inequality for the k^{th} representative micro-structure. First we introduce the following additional five quantities which we associate with a motion of the micro-structure:

The specific internal energy $\epsilon^* = \epsilon^*(\theta^\alpha, \theta^{3(k)}, \xi, t)$

The heat flux vector $q^* = q^*(\theta^\alpha, \theta^{3(k)}, \xi, t)$

The heat supply or heat absorption $r^* = r^*(\theta^\alpha, \theta^{3(k)}, \xi, t)$

The specific entropy $\eta^* = \eta^*(\theta^\alpha, \theta^{3(k)}, \xi, t)$ and,

The local temperature $\theta^* = \theta^*(\theta^\alpha, \theta^{3(k)}, \xi, t)$ which is assumed to be always positive. The equation for the local balance of energy — the first law of thermodynamics — can be written in the following form

$$\rho^* r^* - \rho^* \dot{\epsilon}^* + \tau^{*ij} \dot{\gamma}_{ij}^* - q^{*k}{}_{|k} = 0 \quad (6.1)$$

where ρ^* is the density of the micro-structure, q^{*k} and $\dot{\gamma}_{ij}^*$ are defined by

$$q^* = q^{*k} g_k^* \quad , \quad \dot{\gamma}_{ij}^* = \frac{1}{2} \dot{g}_{ij}^* \quad (6.2)$$

and covariant differentiation is performed with respect to the metric tensor g_{ij}^* of the micro-structure. Recalling the relations

$$\begin{aligned} g_i^* &= v_{,i}^* \\ g_{ij}^* &= g_i^* \cdot g_j^* \end{aligned} \quad (6.3)$$

$$\mathbf{T}^{*i} = g^{*1/2} \tau^{*ij} g_j^*$$

We can write

$$\dot{\gamma}_{ij}^* = \frac{1}{2} \dot{g}_{ij}^* = \frac{1}{2} (v_{,i}^* \cdot g_j^* + v_{,j}^* \cdot g_i^*) \quad (6.4)$$

Using (6.3)₃ and the symmetry of τ^{*ij} , we can write

$$\begin{aligned} \tau^{*ij} \dot{\gamma}_{ij}^* &= \frac{1}{2} (v_{,i}^* \cdot \tau^{*ij} g_j^* + v_{,j}^* \cdot \tau^{*ij} g_i^*) \\ &= \frac{1}{2} (v_{,i}^* \cdot g^{*-1/2} \mathbf{T}^{*i} + v_{,j}^* \cdot g^{*-1/2} \mathbf{T}^{*j}) = g^{*-1/2} \mathbf{T}^{*i} \cdot v_{,i}^* \end{aligned} \quad (6.5)$$

As for the divergence of the heat flux vector, we have

$$\text{div } \mathbf{q}^* = q^{*k}{}_{,k} = \frac{1}{g^{*1/2}} (g^{*1/2} q^{*k})_{,k} \quad (6.6)$$

Introducing the results (6.5) and (6.6) in (6.1), we can write the local energy equation in the following alternative form

$$\rho^* \dot{r}^* - \rho^* \dot{\epsilon}^* + g^{*-1/2} [\mathbf{T}^{*i} \cdot v_{,i}^* - (g^{*1/2} q^{*k})_{,k}] = 0 \quad (6.7)$$

The energy equation can also be written in terms of the Helmholtz free energy function defined by

$$\psi^* = \epsilon^* - \theta^* \eta^* \quad (6.8)$$

The Clausius-Duhem inequality as a statement for second law of thermodynamics has the following local form for the representative micro-structure

$$\rho^* \theta^* \dot{\eta}^* - \rho^* \dot{r}^* + \theta^* g^{*-1/2} \left(\frac{g^{*1/2} q^{*k}}{\theta^*} \right)_{,k} \geq 0 \quad (6.9)$$

By combining (6.9) and (6.1) and using (6.6) we have the inequality

$$\rho^*(\theta^*\dot{\eta}^* - \dot{\epsilon}^*) + \tau^{*ij}\dot{\gamma}_{ij}^* - \frac{1}{\theta^*} q^{*k}\theta_{,k}^* \geq 0 \quad (6.10)$$

which in terms of the Helmholtz free energy ψ^* defined in (6.8) becomes

$$-\rho^*(\psi^* + \eta^*\dot{\theta}^*) + \tau^{*ij}\dot{\gamma}_{ij}^* - \frac{1}{\theta^*} q^{*k}\theta_{,k}^* \geq 0 \quad (6.11)$$

Now for elastic materials the constitutive relations for Helmholtz free energy, the specific entropy and the stress tensor can be expressed in the following forms

$$\psi^* = \tilde{\psi}^*(\gamma_{ij}^*, \theta^*) \quad (6.12)$$

$$\eta^* = - \frac{\partial \tilde{\psi}^*}{\partial \theta^*} \quad (6.13)$$

$$\tau^{*ij} = \rho^* \frac{\partial \tilde{\psi}^*}{\partial \gamma_{ij}^*} \quad (6.14)$$

where the partial derivative with respect to the symmetric tensor γ_{ij}^* is understood to have the following symmetric form

$$\frac{1}{2} \left(\frac{\partial \tilde{\psi}^*}{\partial \gamma_{ij}^*} + \frac{\partial \tilde{\psi}^*}{\partial \gamma_{ji}^*} \right)$$

The constitutive relation for the heat flux vector has the form

$$q^{*k} = \tilde{q}^{*k}(\gamma_{ij}^*, \theta^*, \theta_{,m}^*) \quad (6.15)$$

and the response function \tilde{q}^{*k} in the light of the Clausius-Duhem inequality is seen to be restricted by the inequality

$$-\tilde{q}^{*k}\theta_{,k}^* \geq 0 \quad (6.16)$$

With the help of (6.13) and (6.14) the energy equation (6.1) is reduced to the following form

$$\rho^* \dot{r}^* - q^{*k}_{|k} - \rho^* \theta^* \dot{\eta}^* = 0 \quad (6.17)$$

where we have used the definition (6.8) in order to calculate $\dot{\epsilon}^*$ in terms of $\dot{\psi}^*$ and then used the relations (6.13) and (6.14) to further simplify the energy equation. It should be recalled that the argument of different functions in the energy equation (6.17) is $(\theta^\alpha, \theta^{3(k)}, \xi, t)$ and this equation is written for each and every representative element ($k = 1, 2, \dots, n$) which repeats itself in our model and $n \rightarrow \infty$. For a bi-laminate representative micro-structure with thickness ξ_2 we introduce the following composite quantities. These relations can be generalized for a multi-constituent micro-structure without any difficulty (see definitions 5.3 and 5.4)

$$\rho g^{1/2} \tau \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} \tau^* d\xi \quad (6.18)$$

$$\rho g^{1/2} \tau_1 \triangleq \int_0^{\xi_2} \rho^* g^{*1/2} \tau^* \xi d\xi$$

$$g^{1/2} q^i \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} g^{*1/2} q^{*i} d\xi \quad (6.19)$$

$$g^{1/2} q_1^\alpha \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} g^{*1/2} q^{*\alpha} \xi d\xi$$

$$\rho g^{1/2} \eta_{(m)} \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} \eta^* \xi^m d\xi \quad (m = 0, 1, 2) \quad (6.20)$$

We further assume that the variation of temperature θ^* across the micro-structure is a linear function of ξ , hence

$$\theta^*(\theta^\alpha, \theta^{3(k)}, \xi, t) = \phi_0(\theta^\alpha, \theta^{3(k)}, t) + \xi \phi_1(\theta^\alpha, \theta^{3(k)}, t) \quad (6.21)$$

In order to derive the appropriate form of the energy equation for the composite laminate, first we write (6.17) in the following form

$$\rho^* g^{*1/2} \dot{\tau}^* - (g^{*1/2} q^{*k})_{,k} = \rho^* g^{*1/2} \dot{\eta}^* \theta^* \quad (6.22)$$

which after using (6.21) reduces to

$$\rho^* g^{*1/2} \dot{\tau}^* - (g^{*1/2} q^{*k})_{,k} = \rho^* g^{*1/2} \dot{\eta}^* (\phi_0 + \xi \phi_1) \quad (6.23)$$

Now divide (6.23) by ξ_2 and integrate with respect to ξ from 0 to ξ_2 , the result is

$$\begin{aligned} \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} \dot{\tau}^* d\xi - \frac{1}{\xi_2} \int_0^{\xi_2} (g^{*1/2} q^{*\alpha})_{,\alpha} d\xi - \frac{1}{\xi_2} \int_0^{\xi_2} \frac{\partial}{\partial \xi} (g^{*1/2} q^{*3}) d\xi \\ = \frac{\phi_0}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} \dot{\eta}^* d\xi + \frac{\phi_1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} \dot{\eta}^* \xi d\xi \end{aligned} \quad (6.24)$$

Each term in the above equation can be written in terms of the composite quantities introduced in (6.18)-(6.20) except the third term which is the difference between the values of $g^{*1/2} q^{*3}$ above and below the representative element divided by its thickness ξ_2 , namely

$$\frac{1}{\xi_2} \int_0^{\xi_2} \frac{\partial}{\partial \xi} (g^{*1/2} q^{*3}) d\xi = \frac{1}{\xi_2} [g^{*1/2} q^{*3}(\theta^\alpha, \theta^{3(k+1)}, t) - g^{*1/2} q^{*3}(\theta^\alpha, \theta^{3(k)}, t)]$$

Now we assume the existence of the continuous function $h(\theta^\alpha, \theta^3, t)$ which coincides with $q^{*3}(\theta^\alpha, \theta^{3(k)}, t)$ at $\theta^3 = \theta^{3(k)}$, and further approximate the right side of the above equation as the gradient of this function multiplied by $g^{1/2}$ in the θ^3 direction, i.e., $\partial(g^{1/2} h)/\partial \theta^3$. As a result, (6.24) can be written as

$$\rho g^{1/2} \dot{\tau} - (g^{1/2} q^\alpha)_{,\alpha} - \frac{\partial}{\partial \theta^3} (g^{1/2} h) = \rho g^{1/2} (\phi_0 \dot{\eta}_0 + \phi_1 \dot{\eta}_1) \quad (6.25)$$

In writing (6.25) we have also made use of the balance of mass equation.

Next we multiply (6.23) by ξ , integrate with respect to ξ from 0 to ξ_2 and divide it by ξ_2 to get

$$\begin{aligned} & \frac{1}{\xi_2} \left[\int_0^{\xi_2} \rho^* g^{*1/2} r^* \xi \, d\xi - \int_0^{\xi_2} \xi (g^{*1/2} q^{*\alpha})_{,\alpha} - \int_0^{\xi_2} \xi \frac{\partial}{\partial \xi} (g^{*1/2} q^{*3}) d\xi \right] \\ &= \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} \dot{\eta}^* \xi (\phi_0 + \xi \phi_1) d\xi \end{aligned} \quad (6.26)$$

Using integration by part, the third term on the left-hand side of (6.26) can be written as

$$\frac{1}{\xi_2} \int_0^{\xi_2} \xi \frac{\partial}{\partial \xi} (g^{*1/2} q^{*3}) d\xi = \frac{1}{\xi_2} [\xi g^{*1/2} q^{*3}]_0^{\xi_2} - \frac{1}{\xi_2} \int_0^{\xi_2} g^{*1/2} q^{*3} d\xi = g^{1/2} (h - q^3) \quad (6.27)$$

which in writing the last term we have used (6.19)₁ and the definition of h given above. Using this result together with the relations (6.18)₂, (6.19)₂ and (6.20) we can write (6.26) in the following form

$$\rho g^{1/2} r_1 - (g^{1/2} q_1^\alpha)_{,\alpha} - g^{1/2} (h - q^3) = \rho g^{1/2} (\phi_0 \dot{\eta}_1 + \phi_1 \dot{\eta}_2) \quad (6.28)$$

To determine the appropriate form of constraints on the composite heat flux vectors, first we write the Clausius-Duhem inequality (6.16) in the following form

$$g^{*1/2} \bar{q}^{*k} \theta_{,k}^* \leq 0$$

which by (6.21) reduces to

$$g^{*1/2} \bar{q}^{*\alpha} (\phi_{0,\alpha} + \xi \phi_{1,\alpha}) + g^{*1/2} \bar{q}^{*3} \phi_1 \leq 0 \quad (6.29)$$

Next we divide (6.29) by ξ_2 and integrate with respect to ξ from 0 to ξ_2 which after using (6.19) can be written as

$$g^{1/2} \bar{q}^\alpha \phi_{0,\alpha} + g^{1/2} \bar{q}_1^\alpha \phi_{1,\alpha} + g^{1/2} \bar{q}^3 \phi_1 \leq 0$$

or

$$q^\alpha \phi_{0,\alpha} + q_1^\alpha \phi_{1,\alpha} + q^3 \phi_1 \leq 0 \quad (6.30)$$

which is the appropriate form of Clausius-Duhem inequality for the elastic composite laminates.

When the rate of heat supply or absorption is zero ($r = r^1 = 0$) the energy equations for the composite reduce to

$$(g^{1/2} q^\alpha)_{,\alpha} + \frac{\partial}{\partial \theta^3} (g^{1/2} h) + \rho g^{1/2} (\phi_0 \dot{\eta}_0 + \phi_1 \dot{\eta}_1) = 0 \quad (6.31)$$

$$(g^{1/2} q_1^\alpha)_{,\alpha} + g^{1/2} (h - q^3) + \rho g^{1/2} (\phi_0 \dot{\eta}_1 + \phi_1 \dot{\eta}_2) = 0 \quad (6.32)$$

For small deformations of composites with initially flat plies the energy equations (6.31) and (6.32) further reduce to

$$q_{,\alpha}^\alpha + \frac{\partial h}{\partial \theta^3} + \rho(\phi_0 \dot{\eta}_0 + \phi_1 \dot{\eta}_1) = 0 \quad (6.33)$$

$$q_{1,\alpha}^\alpha + h - q^3 + \rho(\phi_0 \dot{\eta}_1 + \phi_1 \dot{\eta}_2) = 0$$

where obviously no distinction should be made between contravariant and covariant components of heat flux vectors.

The derivation of energy equations (6.25)-(6.28) and the Clausius-Duhem inequality (6.30) for the composite laminates is not affected by the number of layers (or constituents) in the representative micro-structure. The only necessary modification in the case of a multi-constituent composite is the replacement of ξ_2 by ξ_n in definitions (6.18)-(6.20). Here ξ_n is the thickness of the representative element which is supposed to consist of n layers. Of course ξ_n is still supposed to be a very small number.

To recapitulate, for a composite whose micro-structure is composed of n layers with a total thickness of ξ_n we have the following relations for energy balance and the Clausius-Duhem inequality

$$\rho r - g^{-1/2}[(g^{1/2}q^\alpha)_{,\alpha} + \frac{\partial}{\partial \theta^3} (g^{1/2} h)] = \rho(\phi_0 \dot{\eta}_0 + \phi_1 \dot{\eta}_1)$$

$$\rho r_1 + q^3 - h - g^{-1/2}(g^{1/2}q_1^\alpha)_{,\alpha} = \rho(\phi_0 \dot{\eta}_1 + \phi_1 \dot{\eta}_2) \quad (6.34)$$

$$q^\alpha \phi_{0,\alpha} + q_1^\alpha \phi_{1,\alpha} + q^3 \phi_1 \geq 0$$

where the composite quantities are defined as follows

$$\rho g^{1/2} r = \frac{1}{\xi_n} \int_0^{\xi_n} \rho^* g^{*1/2} r^* d\xi$$

$$\rho g^{1/2} r_1 = \frac{1}{\xi_n} \int_0^{\xi_n} \rho^* g^{*1/2} r^* \xi d\xi$$

$$\rho g^{1/2} \eta_{(m)} = \frac{1}{\xi_n} \int_0^{\xi_n} \rho^* g^{*1/2} \eta^* \xi^m d\xi \quad (m = 0, 1, 2) \quad (6.35)$$

$$g^{1/2} q^i = \frac{1}{\xi_n} \int_0^{\xi_n} g^{*1/2} q^{*i} d\xi$$

$$g^{1/2} q_1^\alpha = \frac{1}{\xi_n} \int_0^{\xi_n} g^{*1/2} q^{*\alpha} \xi d\xi$$

7.0 CONSTITUTIVE RELATIONS FOR LINEAR THERMO-ELASTICITY

For a composite laminate whose micro-structure is composed of n layers with different linear thermo-elastic constituents, we recall the following constitutive equations for the stress tensor τ^{*ij} , entropy η^* and the heat flux vector q^*

$$\tau_{(\alpha)}^{*ij} = c_{(\alpha)}^{ijkl} \gamma_{kl}^* - c_{(\alpha)}^{ij} \theta^* \quad (7.1)$$

$$(\rho^* \eta^*)_{(\alpha)} = c_{(\alpha)}^{ij} \gamma_{ij}^* + (\rho^* c)_{(\alpha)} \theta^* \quad (7.2)$$

$$q_{(\alpha)}^{*i} = -k_{(\alpha)}^{ij} \theta_{,j}^* \quad (7.3)$$

where $c_{(\alpha)}^{ijkl}$, $c_{(\alpha)}^{ij}$, $c_{(\alpha)}$ and $k_{(\alpha)}^{ij}$ ($\alpha = 1, 2, \dots, n$) are constants in the associated layers. Moreover, we have the following symmetries

$$c_{(\alpha)}^{ijkl} = c_{(\alpha)}^{jikl} = c_{(\alpha)}^{ijlk} = c_{(\alpha)}^{klij} \quad (7.4)$$

$$c_{(\alpha)}^{ij} = c_{(\alpha)}^{ji} \quad (7.5)$$

Now we proceed to calculate the appropriate constitutive relations for *composite stress vector* T^i , *composite couple stress* S^α , *composite entropy* $\eta_{(m)}$ ($m = 0, 1, 2$), and *composite heat flux vectors* q^i and q_1^α . The contribution of the first part of (7.1) to the constitutive relations for T^i and S^α (and consequently τ^{ij} and $S^{\alpha j}$) has already been calculated (see section 5). Therefore we need to find out the effect of the second part of (7.1) in the constitutive relations for T^i and S^α . Similar to what was done in section 5 we adopt the following definitions for the weighted averages of various quantities

$$J^{(k)ij} = \frac{1}{\xi_n} \int_0^{\xi_n} \xi^k c_{(\alpha)}^{ij} d\xi \quad (7.6)$$

$$K^{(k)} = \frac{1}{\xi_n} \int_0^{\xi_n} \xi^k (\rho^* c)_{(\alpha)} d\xi \quad (7.7)$$

$$L^{(k)ij} = \frac{1}{\xi_n} \int_0^{\xi_n} \xi^k k_{(\alpha)}^{ij} d\xi \quad (7.8)$$

Now recalling (5.11), (5.12), (6.21) and (7.6) we can write

$$\begin{aligned} \frac{1}{\xi_n} \int_0^{\xi_n} c_{(\alpha)}^{ij} \theta^* g^{*1/2} g_j^* d\xi &= \frac{\phi_0}{\xi_n} \int_0^{\xi_n} c_{(\alpha)}^{ij} g^{*1/2} g_j^* d\xi \\ &+ \frac{\phi_1}{\xi_n} \int_0^{\xi_n} \xi c_{(\alpha)}^{ij} g^{*1/2} g_j^* d\xi = \frac{\phi_0}{\xi_n} \int_0^{\xi_n} (c^{i\beta} g_\beta^* + c^{i3} g_3^*) g^{*1/2} d\xi \\ &+ \frac{\phi_1}{\xi_n} \int_0^{\xi_n} \xi g^{*1/2} (c^{i\beta} g_\beta^* + c^{i3} g_3^*) d\xi \\ &= \phi_0 g^{1/2} g_\beta \frac{1}{\xi_n} \int_0^{\xi_n} (1 + \frac{\xi \Delta}{2g}) c^{i\beta} d\xi + \phi_0 g^{1/2} \lambda_\beta^j g_j \frac{1}{\xi_n} \int_0^{\xi_n} \xi (1 + \frac{\xi \Delta}{2g}) c^{i\beta} d\xi \\ &+ \phi_0 g^{1/2} g_3 \frac{1}{\xi_n} \int_0^{\xi_n} (1 + \frac{\xi \Delta}{2g}) c^{i3} d\xi + \frac{\phi_1 g^{1/2}}{\xi_n} g_\beta \int_0^{\xi_n} \xi (1 + \frac{\xi \Delta}{2g}) c^{i\beta} d\xi \\ &+ \phi_1 g^{1/2} \lambda_\beta^j g_j \frac{1}{\xi_n} \int_0^{\xi_n} \xi^2 (1 + \frac{\xi \Delta}{2g}) c^{i\beta} d\xi + \frac{\phi_1 g^{1/2} g_3}{\xi_n} \int_0^{\xi_n} \xi (1 + \frac{\xi \Delta}{2g}) c^{i3} d\xi \\ &= \phi_0 g^{1/2} g_j \{ \frac{1}{\xi_n} \int_0^{\xi_n} (1 + \frac{\xi \Delta}{2g}) c^{ij} d\xi + \lambda_\beta^j \frac{1}{\xi_n} \int_0^{\xi_n} \xi (1 + \frac{\xi \Delta}{2g}) c^{i\beta} d\xi \} \\ &+ \phi_1 g^{1/2} g_j \{ \frac{1}{\xi_n} \int_0^{\xi_n} \xi (1 + \frac{\xi \Delta}{2g}) c^{ij} d\xi + \lambda_\beta^j \frac{1}{\xi_n} \int_0^{\xi_n} \xi^2 (1 + \frac{\xi \Delta}{2g}) c^{i\beta} d\xi \} \\ &= \phi_0 g^{1/2} g_j \{ J^{(0)ij} + \frac{\Delta}{2g} J^{(1)ij} + \lambda_\beta^j (J^{(1)i\beta} + \frac{\Delta}{2g} J^{(2)i\beta}) \} \\ &+ \phi_1 g^{1/2} g_j \{ J^{(1)ij} + \frac{\Delta}{2g} J^{(2)ij} + \lambda_\beta^j (J^{(2)i\beta} + \frac{\Delta}{2g} J^{(3)i\beta}) \} \end{aligned} \quad (7.9)$$

By combining the results (7.9) and (5.17) we can obtain the response function for the composite stress vector T^i . As before if we disregard the factor $g^{1/2} g_j$, what remains is the constitutive relation for τ^{ij} which is recorded below

$$\begin{aligned} \tau^{ij} = & \{ I^{(0)ijk} + \frac{\Delta}{2g} I^{(1)ijk} + \lambda_{\beta}^j (I^{(1)i\beta k} + \frac{\Delta}{2g} I^{(2)i\beta k}) \} \gamma_{kl} \\ & + \{ I^{(1)ij\alpha} + \frac{\Delta}{2g} I^{(2)ij\alpha} + \lambda_{\beta}^j (I^{(2)i\beta\alpha} + \frac{\Delta}{2g} I^{(3)i\beta\alpha}) \} \kappa_{l\alpha} \\ & - \phi_0 \{ J^{(0)ij} + \frac{\Delta}{2g} J^{(1)ij} + \lambda_{\beta}^j (J^{(1)i\beta} + \frac{\Delta}{2g} J^{(2)i\beta}) \} \\ & - \phi_1 \{ J^{(1)ij} + \frac{\Delta}{2g} J^{(2)ij} + \lambda_{\beta}^j (J^{(2)i\beta} + \frac{\Delta}{2g} J^{(3)i\beta}) \} \end{aligned} \quad (7.10)$$

Similar steps are followed to find the constitutive equations for S^{α} and $S^{\alpha j}$. The contribution of the thermal term is

$$\begin{aligned} \frac{1}{\xi_n} \int_0^{\xi_n} \xi c_{(\alpha)}^{\alpha j} \theta^* g^{*1/2} g_j^* d\xi = & \phi_0 g^{1/2} g_j \{ J^{(1)\alpha j} + \frac{\Delta}{2g} J^{(2)\alpha j} + \lambda_{\beta}^j (J^{(2)\alpha\beta} + \frac{\Delta}{2g} J^{(3)\alpha\beta}) \} \\ & + \phi_1 g^{1/2} g_j \{ J^{(2)\alpha j} + \frac{\Delta}{2g} J^{(3)\alpha j} + \lambda_{\beta}^j (J^{(3)\alpha\beta} + \frac{\Delta}{2g} J^{(4)\alpha\beta}) \} \end{aligned} \quad (7.11)$$

Again combining (7.11) and (5.20) we find the constitutive relation for S^{α} . Dropping the common factor $g^{1/2} g_j$ would result in the constitutive relation for $S^{\alpha j}$ which is recorded below.

$$\begin{aligned} S^{\alpha j} = & \gamma_{kl} \{ I^{(1)\alpha jkl} + \frac{\Delta}{2g} I^{(2)\alpha jkl} + \lambda_{\gamma}^j (I^{(2)\alpha\gamma k} + \frac{\Delta}{2g} I^{(3)\alpha\gamma k}) \} \\ & + \kappa_{\beta} \{ I^{(2)\alpha j\beta} + \frac{\Delta}{2g} I^{(3)\alpha j\beta} + \lambda_{\gamma}^j (I^{(3)\alpha\gamma\beta} + \frac{\Delta}{2g} I^{(4)\alpha\gamma\beta}) \} \\ & - \phi_0 \{ J^{(1)\alpha j} + \frac{\Delta}{2g} J^{(2)\alpha j} + \lambda_{\beta}^j (J^{(1)\alpha\beta} + \frac{\Delta}{2g} J^{(3)\alpha\beta}) \} \end{aligned}$$

$$- \phi_1 \{ J^{(2)\alpha j} + \frac{\Delta}{2g} J^{(3)\alpha j} + \lambda_{\beta}^j (J^{(3)\alpha\beta} + \frac{\Delta}{2g} J^{(4)\alpha\beta}) \} \quad (7.12)$$

The next step is to find the constitutive relation for the specific entropies $\eta_{(m)}$ ($m = 0, 1, 2$). We calculate the contribution of each term of the relation (7.2) separately. By (6.35)₃ we have

$$\begin{aligned} \rho g^{1/2} \eta_{(m)} &= \frac{1}{\xi_n} \int_0^{\xi_n} \rho^* \eta^* g^{*1/2} \xi^m d\xi \\ &= \frac{1}{\xi_n} \int_0^{\xi_n} c_{(\alpha)}^{ij} \gamma_{ij}^* g^{*1/2} \xi^m d\xi + \frac{1}{\xi_n} \int_{\xi_1}^{\xi_2} (\rho^* c)_{(\alpha)} \theta^* g^{*1/2} \xi^m d\xi \end{aligned} \quad (7.13)$$

Since $g^{*1/2} = g^{1/2} (1 + \frac{\xi\Delta}{2g})$ we can write the first part of (7.13) as

$$\frac{1}{\xi_n} \int_0^{\xi_n} c_{(\alpha)}^{ij} \gamma_{ij}^* g^{*1/2} \xi^m d\xi = \frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi^m (1 + \frac{\xi\Delta}{2g}) c_{(\alpha)}^{ij} \gamma_{ij}^* d\xi \quad (7.14)$$

which by (2.31)-(2.34) reduces to

$$\begin{aligned} &= [\frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi^m (1 + \frac{\xi\Delta}{2g}) c_{(\alpha)}^{ij} d\xi] \gamma_{ij} + [\frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi^{m+1} (1 + \frac{\xi\Delta}{2g}) c_{(\alpha)}^{i\beta} d\xi] \kappa_{i\beta} \\ &= g^{1/2} \{ \gamma_{ij} [J^{(m)ij} + \frac{\Delta}{2g} J^{(m+1)ij}] + \kappa_{i\beta} [J^{(m+1)i\beta} + \frac{\Delta}{2g} J^{(m+2)i\beta}] \} \end{aligned} \quad (7.15)$$

The second part of (7.13), by (6.21) and (7.7), is written as

$$\begin{aligned} \frac{1}{\xi_n} \int_0^{\xi_n} (\rho^* c)_{(\alpha)} \theta^* g^{*1/2} \xi^m d\xi &= \frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi^m (1 + \frac{\xi\Delta}{2g}) (\phi_0 + \xi \phi_1) (\rho^* c)_{(\alpha)} d\xi \\ &= \frac{\phi_0 g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi^m (1 + \frac{\xi\Delta}{2g}) (\rho^* c)_{(\alpha)} d\xi \\ &\quad + \frac{\phi_1 g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi^{m+1} (1 + \frac{\xi\Delta}{2g}) (\rho^* c)_{(\alpha)} d\xi \end{aligned}$$

$$= \phi_0 g^{1/2} \{K^{(m)}\}$$

$$+ \frac{\Delta}{2g} K^{(m+1)}\} + \phi_1 g^{1/2} \{K^{(m+1)} + \frac{\Delta}{2g} K^{(m+2)}\} \quad (7.16)$$

Substituting (7.15) and (7.16) in (7.13) we obtain

$$\begin{aligned} \rho g^{1/2} \eta_{(m)} = & g^{1/2} \{ \gamma_{ij} (J^{(m)ij} + \frac{\Delta}{2g} J^{(m+1)ij}) \} + g^{1/2} \kappa_{i\beta} (J^{(m+1)i\beta} + \frac{\Delta}{2g} J^{(m+2)i\beta}) \\ & + \phi_0 g^{1/2} (K^{(m)} + \frac{\Delta}{2g} K^{(m+1)}) + \phi_1 g^{1/2} (K^{(m+1)} + \frac{\Delta}{2g} K^{(m+2)}) \end{aligned}$$

or

$$\begin{aligned} \rho \eta_{(m)} = & \gamma_{ij} (J^{(m)ij} + \frac{\Delta}{2g} J^{(m+1)ij}) + \kappa_{i\beta} (J^{(m+1)i\beta} + \frac{\Delta}{2g} J^{(m+2)i\beta}) \\ & + \phi_0 (K^{(m)} + \frac{\Delta}{2g} K^{(m+1)}) + \phi_1 (K^{(m+1)} + \frac{\Delta}{2g} K^{(m+2)}) \quad (m = 0, 1, 2) \quad (7.17) \end{aligned}$$

We start with (6.35)₄ to find the constitutive relation for q^i . Substituting from (7.3) in (6.35)₄ we get

$$\begin{aligned} g^{1/2} q^i = & \frac{1}{\xi_n} \int_0^{\xi_n} g^{*1/2} q^{*i} d\xi = - \frac{1}{\xi_n} \int_0^{\xi_2} g^{*1/2} k_{(\alpha)}^{ij} \theta_j^* d\xi \\ = & - \frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} (1 + \frac{\xi \Delta}{2g}) (k^{i\beta} \theta_{,\beta}^* + k^{i3} \theta_{,3}^*) d\xi \end{aligned} \quad (7.18)$$

By (6.21)

$$\theta_{,\beta}^* = \phi_{0,\beta} + \xi \phi_{1,\beta} \quad (7.19)$$

$$\theta_{,3}^* = \phi_1 \quad (7.20)$$

Noting these results we simplify (7.18). Hence

$$\begin{aligned}
 g^{1/2} q^i &= - \frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} \left(1 + \frac{\xi \Delta}{2g}\right) k^{i\beta} (\phi_{0,\beta} + \xi \phi_{1,\beta}) d\xi \\
 &\quad - \frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} \left(1 + \frac{\xi \Delta}{2g}\right) k^{i3} \phi_1 d\xi \\
 &= - g^{1/2} \left\{ \frac{\phi_{0,\beta}}{\xi_n} \int_0^{\xi_n} \left(1 + \frac{\xi \Delta}{2g}\right) k^{i\beta} d\xi + \frac{\phi_{1,\beta}}{\xi_n} \int_0^{\xi_n} \xi \left(1 + \frac{\xi \Delta}{2g}\right) k^{i\beta} d\xi \right. \\
 &\quad \left. + \frac{\phi_1}{\xi_n} \int_0^{\xi_n} \left(1 + \frac{\xi \Delta}{2g}\right) k^{i3} d\xi \right\} \tag{7.21}
 \end{aligned}$$

Using (7.8) we get

$$\begin{aligned}
 g^{1/2} q^i &= - g^{1/2} \left\{ \phi_{0,\beta} [L^{(0)i\beta} + \frac{\Delta}{2g} L^{(1)i\beta}] + \phi_{1,\beta} [L^{(1)i\beta} + \frac{\Delta}{2g} L^{(2)i\beta}] \right. \\
 &\quad \left. + \phi_1 [L^{(0)i3} + \frac{\Delta}{2g} L^{(1)i3}] \right\}
 \end{aligned}$$

or

$$\begin{aligned}
 q^i &= - (L^{(0)i\beta} + \frac{\Delta}{2g} L^{(1)i\beta}) \phi_{0,\beta} - (L^{(1)i\beta} + \frac{\Delta}{2g} L^{(2)i\beta}) \phi_{1,\beta} \\
 &\quad - (L^{(0)i3} + \frac{\Delta}{2g} L^{(1)i3}) \phi_1 \tag{7.22}
 \end{aligned}$$

Finally we use (6.35)₅ to derive the constitutive equation for q_1^α . Similar to the above development, we write

$$g^{1/2} q_1^\alpha = \frac{1}{\xi_n} \int_0^{\xi_n} g^{*1/2} q^{*\alpha} \xi d\xi = - \frac{1}{\xi_n} \int_0^{\xi_n} \xi g^{*1/2} k^{\alpha j} \theta_{,j}^* d\xi$$

$$\begin{aligned}
 &= -\frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi \left(1 + \frac{\xi \Delta}{2g}\right) (k^{\alpha\beta} \theta_{,\beta}^* + k^{\alpha 3} \theta_{,3}^*) d\xi \\
 &= -\frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi \left(1 + \frac{\xi \Delta}{2g}\right) k^{\alpha\beta} (\phi_{o,\beta} + \xi \phi_{1,\beta}) d\xi - \frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi \left(1 + \frac{\xi \Delta}{2g}\right) k^{\alpha 3} \phi_1 d\xi \quad (7.23)
 \end{aligned}$$

which by (7.8) reduces to

$$q_1^\alpha = -(L^{(1)\alpha\beta} + \frac{\Delta}{2g} L^{(2)\alpha\beta}) \phi_{o,\beta} - (L^{(2)\alpha\beta} + \frac{\Delta}{2g} L^{(3)\alpha\beta}) \phi_{1,\beta} - (L^{(1)\alpha 3} + \frac{\Delta}{2g} L^{(2)\alpha 3}) \phi_1 \quad (7.24)$$

This concludes our derivation of linear thermo-elastic constitutive relations for composite laminates.

For small deformations of a composite with initially flat plies the foregoing equations are simplified to the following constitutive relations:

$$\tau_{ij} = I_{ijk}^{(0)} u_{k,l} + I_{ijl\alpha}^{(1)} u_{l,\alpha 3} - J_{ij}^{(0)} \phi_o - J_{ij}^{(1)} \phi_1 \quad (7.25)$$

$$S_{\alpha j} = I_{\alpha jkl}^{(1)} u_{k,l} + I_{\alpha j\beta}^{(2)} u_{l,\beta 3} - J_{\alpha j}^{(1)} \phi_o - J_{\alpha j}^{(2)} \phi_1 \quad (7.26)$$

$$\rho \eta_{(m)} = J_{ij}^{(m)} u_{i,j} + J_{i\beta}^{(m+1)} u_{i,\beta 3} + K^{(m)} \phi_o + K^{(m+1)} \phi_1 \quad (m = 0, 1, 2) \quad (7.27)$$

$$q_i = -L_{i\beta}^{(0)} \phi_{o,\beta} - L_{i\beta}^{(1)} \phi_{1,\beta} - L_{i3}^{(0)} \phi_1 \quad (7.28)$$

$$q_1^\alpha = -L^{(1)\alpha\beta} \phi_{o,\beta} - L^{(2)\alpha\beta} \phi_{1,\beta} - L^{(1)\alpha 3} \phi_1 \quad (7.29)$$

The constitutive coefficients $I^{(0)}$, $I^{(1)}$ and $I^{(2)}$ have already been calculated and recorded in equations (5.26). As for the other constitutive coefficients we use the results of section 5. Comparing definitions (7.6)-(7.8) with (5.8) and using the results (5.4) and (5.30) we can write

$$J^{(k)ij} = \frac{1}{k+1} \xi_n^k \sum_{r=1}^n c_{(r)}^{ij} \Delta m_r^{k+1} \quad (7.30)$$

$$K^{(k)} = \frac{1}{k+1} \xi_n^k \sum_{r=1}^n (\rho^* c)_{(r)} \Delta m_r^{k+1} \quad (7.31)$$

$$L^{(k)ij} = \frac{1}{k+1} \xi_n^k \sum_{r=1}^n k_{(r)}^{ij} \Delta m_r^{k+1} \quad (7.32)$$

It should be noted that $k_{(r)}^{ij}$ in equation (7.32) are the coefficients of thermal conductivity of different layers of the representative micro-structure and are not to be confused with the superscript k which assumes non-negative integer values.

If the micro-structure is composed of isotropic layers, the coefficients of thermal stress $c_{(r)}^{ij}$ and thermal conductivity $k_{(r)}^{ij}$ can be written in terms of only one constant for each layer.

For such case we write

$$c_{ij}^{(r)} = \beta_{(r)} \delta_{ij} \quad (7.33)$$

$$k_{ij}^{(r)} = k_{(r)} \delta_{ij} \quad (7.34)$$

Taking note of these relations and relations (7.30) and (7.32) we obtain

$$\begin{aligned} J_{ij}^{(0)} &= \delta_{ij} \sum_{r=1}^n \beta_{(r)} \Delta m_r \\ J_{ij}^{(1)} &= \frac{1}{2} \xi_n \delta_{ij} \sum_{r=1}^n \beta_{(r)} \Delta m_r^2 \\ J_{ij}^{(2)} &= \frac{1}{3} \xi_n^2 \delta_{ij} \sum_{r=1}^n \beta_{(r)} \Delta m_r^3 \\ J_{ij}^{(3)} &= \frac{1}{4} \xi_n^3 \delta_{ij} \sum_{r=1}^n \beta_{(r)} \Delta m_r^4 \\ L_{ij}^{(0)} &= \delta_{ij} \sum_{r=1}^n k_{(r)} \Delta m_r \\ L_{ij}^{(1)} &= \frac{1}{2} \xi_n \delta_{ij} \sum_{r=1}^n k_{(r)} \Delta m_r^2 \end{aligned} \quad (7.35)$$

(7.36)

$$L_{ij}^{(2)} = \frac{1}{3} \xi_n^2 \delta_{ij} \sum_{r=1}^n k_{(r)} \Delta m_r^3$$

Consequently the constitutive equations (7.25)-(7.29) reduce

$$\begin{aligned} \tau_{ij} = & \delta_{ij} \sum_{r=1}^n (\lambda_{(r)} u_{k,k} - \beta_{(r)} \phi_o) \Delta m_r + (u_{i,j} + u_{j,i}) \sum_{r=1}^n \mu_{(r)} \Delta m_r \\ & + \frac{1}{2} \xi_n \delta_{ij} \sum_{r=1}^n (\lambda_{(r)} u_{\alpha,\alpha 3} - \beta_{(r)} \phi_1) \Delta m_r^2 \\ & + \frac{\xi_n}{2} (u_{i,\alpha 3} \delta_{j\alpha} + u_{j,\alpha 3} \delta_{i\alpha}) \sum_{r=1}^n \mu_{(r)} \Delta m_r^2 \end{aligned} \quad (7.37)$$

$$\begin{aligned} S_{\alpha j} = & \frac{1}{2} \xi_n \{ \delta_{\alpha j} \sum_{r=1}^n (\lambda_{(r)} u_{k,k} - \beta_{(r)} \phi_o) \Delta m_r^2 + (u_{\alpha,j} + u_{j,\alpha}) \sum_{r=1}^n \mu_{(r)} \Delta m_r^2 \} \\ & + \frac{1}{3} \xi_n^2 \{ \delta_{\alpha j} \sum_{r=1}^n (\lambda_{(r)} u_{\beta,\beta 3} - \beta_{(r)} \phi_1) \Delta m_r^3 \\ & + (\delta_{j\beta} u_{\alpha,\beta 3} + u_{j,\alpha 3} \sum_{r=1}^n \mu_{(r)} \Delta m_r^3) \} \end{aligned} \quad (7.38)$$

$$\begin{aligned} \rho \eta_{(0)} = & \sum_{r=1}^n [\beta_{(r)} u_{i,i} + (\rho^* c)_{(r)} \phi_o] \Delta m_r \\ & + \frac{1}{2} \xi_n \sum_{r=1}^n [\beta_{(r)} u_{\beta,\beta 3} + (\rho^* c)_{(r)} \phi_1] \Delta m_r^2 \end{aligned} \quad (7.39)$$

$$\begin{aligned} \rho \eta_{(1)} = & \frac{1}{2} \xi_n \sum_{r=1}^n [\beta_{(r)} u_{i,i} + (\rho^* c)_{(r)} \phi_o] \Delta m_r^2 \\ & + \frac{1}{3} \xi_n^2 \sum_{r=1}^n [\beta_{(r)} u_{\beta,\beta 3} + (\rho^* c)_{(r)} \phi_1] \Delta m_r^3 \end{aligned} \quad (7.40)$$

$$\rho \eta_{(2)} = \frac{1}{3} \xi_n^2 \sum_{r=1}^n [\beta_{(r)} u_{i,i} + (\rho^* c)_{(r)} \phi_o] \Delta m_r^3$$

$$+ \frac{1}{4} \xi_n^3 \sum_{r=1}^n [\beta_{(r)} u_{\beta, \beta 3} + (\rho^* c)_{(r)} \phi_1] \Delta m_r^4 \quad (7.41)$$

$$q_\alpha = - \phi_{o, \alpha} \sum_{r=1}^n k_{(r)} \Delta m_r - \frac{1}{2} \xi_n \phi_{1, \alpha} \sum_{r=1}^n k_{(r)} \Delta m_r^2 \quad (7.42)$$

$$q_3 = - \phi_1 \sum_{r=1}^n k_{(r)} \Delta m_r \quad (7.43)$$

$$q_1^\alpha = - \frac{1}{2} \xi_n \phi_{o, \alpha} \sum_{r=1}^n k_{(r)} \Delta m_r^2 - \frac{1}{3} \xi_n^2 \phi_{1, \alpha} \sum_{r=1}^n k_{(r)} \Delta m_r^3 \quad (7.44)$$

In relation (7.44), α is written as a superscript only for convenience and does not signify the contravariant position.

The linear equations of motion and balance of energy for a composite with initially flat plies are derived by substituting (7.25)-(7.29) in (2.123), (2.124) and (6.33). Using the results (5.41) and (5.42) in conjunction with (7.25) and (7.26) we have the following equations of motion in linear thermo-elastic theory

$$\begin{aligned} I_{\alpha j k l}^{(0)} u_{k, \alpha} + I_{\alpha j \beta}^{(1)} u_{l, \alpha \beta 3} - J_{\alpha j}^{(0)} \phi_{0, \alpha} - J_{\alpha j}^{(1)} \phi_{1, \alpha} + b_j \sum_{r=1}^n \rho_o^{(r)} \Delta m_r + \sigma_{j, 3} \\ = \ddot{u}_j \sum_{r=1}^n \rho_o^{(r)} \Delta m_r + \frac{1}{2} \xi_n \ddot{u}_{j, 3} \sum_{r=1}^n \rho_o^{(r)} \Delta m_r^2 \end{aligned} \quad (7.45)$$

$$\begin{aligned} I_{\alpha j k l}^{(1)} u_{k, \alpha} + I_{\alpha j \beta}^{(2)} u_{l, \alpha \beta 3} - J_{\alpha j}^{(1)} \phi_{0, \alpha} - J_{\alpha j}^{(2)} \phi_{1, \alpha} + \sigma_j - I_{3 j k l}^{(0)} u_{k, l} \\ - I_{3 j \beta}^{(1)} u_{l, \beta 3} + J_{3 j}^{(0)} \phi_o + J_{3 j}^{(1)} \phi_1 + c_j \sum_{r=1}^n \rho_o^{(r)} \Delta m_r \\ = \frac{1}{2} \xi_n \ddot{u}_j \sum_{r=1}^n \rho_o^{(r)} \Delta m_r^2 + \frac{1}{3} \xi_n^2 \ddot{u}_{j, 3} \sum_{r=1}^n \rho_o^{(r)} \Delta m_r^3 \end{aligned} \quad (7.46)$$

The energy equations when the rate of heat supply or absorption is zero are recorded in relations (6.33) for small deformations of thermo-elastic composites with initially flat plies.

Substituting the constitutive relations (7.27)-(7.29) in (6.33) we find the following coupled differential equations for displacement and temperature fields

$$(\phi_o J_{ij}^{(0)} + \phi_1 J_{ij}^{(1)}) \dot{u}_{i,j} + (\phi_o J_{i\beta}^{(1)} + \phi_1 J_{i\beta}^{(2)}) \dot{u}_{i,\beta 3} + (\phi_o K^{(0)} + \phi_1 K^{(1)}) \dot{\phi}_o + (\phi_o K^{(1)} + \phi_1 K^{(2)}) \dot{\phi}_1 + \frac{\partial h}{\partial \theta^3} - L_{\alpha\beta}^{(0)} \phi_{0,\alpha\beta} - L_{\alpha\beta}^{(1)} \phi_{1,\alpha\beta} - L_{\alpha 3}^{(0)} \phi_{1,\alpha} = 0 \quad (7.47)$$

$$(\phi_o J_{ij}^{(1)} + \phi_1 J_{ij}^{(2)}) \dot{u}_{i,j} + (\phi_o J_{i\beta}^{(2)} + \phi_1 J_{i\beta}^{(3)}) \dot{u}_{i,\beta 3} + (\phi_o K^{(1)} + \phi_1 K^{(2)}) \dot{\phi}_o + (\phi_o K^{(2)} + \phi_1 K^{(3)}) \dot{\phi}_1 + h + L_{3\beta}^{(0)} \phi_{0,\beta} + L_{3\beta}^{(1)} \phi_{1,\beta} + L_{33}^{(0)} \phi_1 - L_{\alpha\beta}^{(1)} \phi_{0,\alpha\beta} - L_{\alpha\beta}^{(2)} \phi_{1,\alpha\beta} - L_{\alpha 3}^{(1)} \phi_{1,\alpha} = 0 \quad (7.48)$$

For static problems in the absence of body force and heat supply, the foregoing equations are further reduced to

$$I_{\alpha j k l}^{(0)} u_{k,l\alpha} + I_{\alpha j l \beta}^{(1)} u_{l,\beta 3\alpha} - J_{\alpha j}^{(0)} \phi_{0,\alpha} - J_{\alpha j}^{(1)} \phi_{1,\alpha} + \sigma_{j,3} = 0 \quad (7.49)$$

$$I_{\alpha j k l}^{(1)} u_{k,l\alpha} + I_{\alpha j l \beta}^{(2)} u_{l,\alpha\beta 3} - J_{\alpha j}^{(1)} \phi_{0,\alpha} - J_{\alpha j}^{(2)} \phi_{1,\alpha} + \sigma_j - I_{3 j k l}^{(0)} u_{k,l} - I_{3 j l \beta}^{(1)} u_{l,\beta 3} + J_{3 j}^{(0)} \phi_o + J_{3 j}^{(1)} \phi_1 = 0 \quad (7.50)$$

$$\frac{\partial h}{\partial \theta^3} - L_{\alpha\beta}^{(0)} \phi_{0,\alpha\beta} - L_{\alpha\beta}^{(1)} \phi_{1,\alpha\beta} - L_{\alpha 3}^{(0)} \phi_{1,\alpha} = 0 \quad (7.51)$$

$$h + L_{3\beta}^{(0)} \phi_{0,\beta} + L_{3\beta}^{(1)} \phi_{1,\beta} + L_{33}^{(0)} \phi_1 - L_{\alpha\beta}^{(1)} \phi_{0,\alpha\beta} - L_{\alpha\beta}^{(2)} \phi_{1,\alpha\beta} - L_{\alpha 3}^{(1)} \phi_{1,\alpha} = 0 \quad (7.52)$$

Similar to what was done previously in order to find a relation between the director displacement and the gradient of displacement vector, we enforce the continuity of the temperature field across two adjacent micro-structures to derive an analogous relation between ϕ_o and ϕ_1 defined in equation (6.21). In order that the temperature field be continuous on the common surface between k^{th} and $(k+1)^{st}$ micro-structures we should have

$$\theta^*(\theta^\alpha, \theta^{3(k+1)}, 0, t) = \theta^*(\theta^\alpha, \theta^{3(k)}, \xi_n, t) \quad (7.53)$$

Now by (6.21) we have

$$\theta^*(\theta^\alpha, \theta^{3(k+1)}, 0, t) = \phi_o(\theta^\alpha, \theta^{3(k+1)}, t) \quad (7.54)$$

$$\theta^*(\theta^\alpha, \theta^{3(k)}, \xi_n, t) = \phi_o(\theta^\alpha, \theta^{3(k)}, t) + \xi_n \phi_1(\theta^\alpha, \theta^{3(k)}, t) \quad (7.55)$$

Substituting from (7.54) and (7.55) in (7.53) we get

$$\phi_o(\theta^\alpha, \theta^{3(k+1)}, t) = \phi_o(\theta^\alpha, \theta^{3(k)}, t) + \xi_n \phi_1(\theta^\alpha, \theta^{3(k)}, t)$$

or

$$\phi_1(\theta^\alpha, \theta^{3(k)}, t) = \frac{1}{\xi_n} \{ \phi_o(\theta^\alpha, \theta^{3(k+1)}, t) - \phi_o(\theta^\alpha, \theta^{3(k)}, t) \} \quad (7.56)$$

By smoothing assumptions and noting the smallness of ξ_n we approximate the right-hand side of (7.56) as the gradient of ϕ_o in the θ^3 direction. So we obtain

$$\phi_1(\theta^\alpha, \theta^3, t) = \frac{\partial}{\partial \theta^3} \phi_o(\theta^\alpha, \theta^3, t) \quad (7.57)$$

This conclusion is used in various field equations. In particular, equation (7.49)-(7.52) reduce to

$$I_{\alpha j k l}^{(0)} u_{k, l \alpha} + I_{\alpha j \beta}^{(1)} u_{l, \alpha \beta 3} - J_{\alpha j}^{(0)} \phi_{0, \alpha} - J_{\alpha j}^{(1)} \phi_{0, \alpha 3} + \sigma_{j, 3} = 0 \quad (7.58)$$

$$\begin{aligned} I_{\alpha j k l}^{(1)} u_{k, l \alpha} + I_{\alpha j \beta}^{(2)} u_{l, \alpha \beta 3} - J_{\alpha j}^{(1)} \phi_{0, \alpha} - J_{\alpha j}^{(2)} \phi_{0, \alpha 3} + \sigma_j - I_{3 j k l}^{(0)} u_{k, l} \\ - I_{3 j \beta}^{(1)} u_{l, \beta 3} + J_{3 j}^{(0)} \phi_o + J_{3 j}^{(1)} \phi_{0, 3} = 0 \end{aligned} \quad (7.59)$$

$$h_{, 3} - L_{\alpha \beta}^{(0)} \phi_{0, \alpha \beta} - L_{\alpha \beta}^{(1)} \phi_{0, \alpha \beta 3} - L_{\alpha 3}^{(0)} \phi_{0, \alpha 3} = 0 \quad (7.60)$$

$$h + L_{3 \beta}^{(0)} \phi_{0, \beta} + L_{3 \beta}^{(1)} \phi_{0, \beta 3} + L_{3 3}^{(0)} \phi_{0, 3} - L_{\alpha \beta}^{(1)} \phi_{0, \alpha \beta} - L_{\alpha \beta}^{(2)} \phi_{0, \alpha \beta 3} - L_{\alpha 3}^{(1)} \phi_{0, \alpha 3} = 0 \quad (7.61)$$

Eliminating σ_j between (7.58) and (7.59), and h between (7.60) and (7.61) we find the following coupled differential equations for displacement and temperature fields. Since we are investigating the static problems in the present derivation the equation for temperature, i.e., the equation resulting from the energy equations is independent of the displacement field. Recalling (5.45), the displacement equation becomes

$$\begin{aligned} I_{ijk}^{(0)} u_{k,i} + I_{ijk}^{(1)} u_{k,i\beta 3} - I_{\alpha jk}^{(1)} u_{k,l\alpha 3} - I_{\alpha jk}^{(2)} u_{k,\alpha\beta 33} \\ - J_{\alpha i}^{(0)} \phi_{0,i} - J_{ij}^{(1)} \phi_{0,i3} + J_{\alpha j}^{(1)} \phi_{0,\alpha 3} + J_{\alpha j}^{(2)} \phi_{0,\alpha\beta 33} = 0 \end{aligned} \quad (7.62)$$

and the temperature equation by (7.60) and (7.61) is

$$L_{ij}^{(0)} \phi_{0,ij} + (L_{3\alpha}^{(1)} - L_{\alpha 3}^{(1)}) \phi_{0,\alpha\beta 33} - L_{\alpha\beta}^{(2)} \phi_{0,\alpha\beta 33} = 0 \quad (7.63)$$

Having determined the displacement and the temperature fields, the interlaminar stresses σ_j and heat flux h can be determined from (7.59) and (7.61), respectively. The results are

$$\begin{aligned} \sigma_j = I_{3jk}^{(0)} u_{k,k} + I_{3jk}^{(1)} u_{k,\beta 3} - I_{\alpha jk}^{(1)} u_{k,l\alpha} - I_{\alpha j\beta}^{(2)} u_{l,\alpha\beta 3} \\ + J_{\alpha j}^{(1)} \phi_{0,\alpha} + J_{\alpha j}^{(2)} \phi_{0,\alpha 3} - J_{3j}^{(0)} \phi_0 - J_{3j}^{(1)} \phi_{0,3} \end{aligned} \quad (7.64)$$

$$\begin{aligned} h = L_{\alpha\beta}^{(1)} \phi_{0,\alpha\beta} + L_{\alpha\beta}^{(2)} \phi_{0,\alpha\beta 3} + L_{\alpha 3}^{(1)} \phi_{0,\alpha 3} - L_{3\alpha}^{(0)} \phi_{0,\alpha} - L_{3\alpha}^{(1)} \phi_{0,\alpha 3} - L_{33}^{(2)} \phi_{0,3} \\ = L_{\alpha\beta}^{(1)} \phi_{0,\alpha\beta} + L_{\alpha\beta}^{(2)} \phi_{0,\alpha\beta 3} + (L_{\alpha 3}^{(1)} - L_{3\alpha}^{(1)}) \phi_{0,\alpha 3} - L_{3j}^{(0)} \phi_{0,j} \end{aligned} \quad (7.65)$$

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